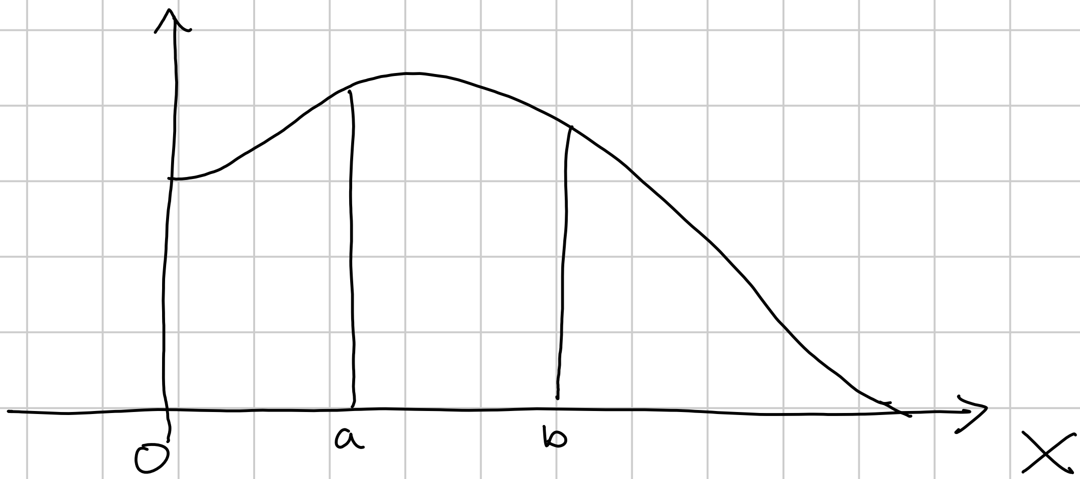
Continuous Probability Distributions

A continuous p.d is represented by a curve rather than a histogram:



The probability that $a < X < b$ is represented by the area under the curve between a and b.

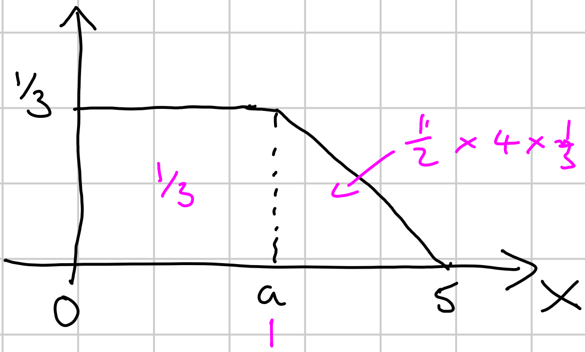
The total area under the curve must be 1.

The equation of the curve $y = f(x)$ is called the probability density function of the random variable X.

We can find probabilities (ie areas) by integrating $f(x)$

Examples

① The r.v. X has p.d.f. represented by the following graph. Find the value of a.



Total area under graph = 1

$$\frac{1}{3}a + \frac{1}{2}(5-a)\frac{1}{3} = 1$$

$(\times 3)$ $$a + \frac{1}{2}(5-a) = 3$$

$(\times 2)$ $$2a + (5-a) = 6$$

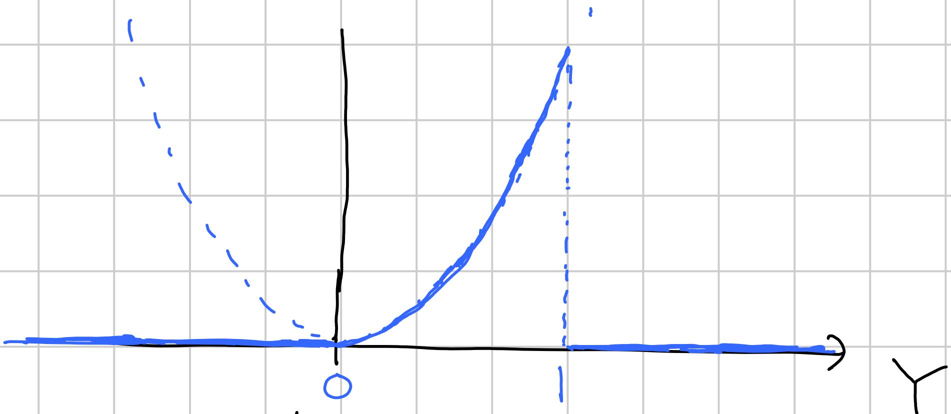
$$a + 5 = 6$$

$$\underline{\underline{a = 1}}$$

② The random variable Y has p.d.f defined as follows :-

$$f(y) = \begin{cases} ky^2 & (0 \leqslant y \leqslant 1) \\ 0 & (\text{otherwise}) \end{cases}$$

Sketch the p.d.f and find the value of k.



$$\int_0^1 ky^2 \, dy = 1$$

$$\left[\frac{k}{3}y^3\right]_0^1 = 1$$

$$\left(\frac{k}{3}\right) - (0) = 1$$

$$\underline{\underline{k = 3}}$$

p 40 Ex 2A Q 1 – 5 by Friday

③ A shot-putter puts the shot a distance $10X$ metres where X is a r.v. with p.d.f. $f(x)$ defined as

$$f(x) = \begin{cases} \frac{3}{4}x(x-2)^2 & (0 \leq x \leq 2) \\ 0 & (\text{otherwise}) \end{cases}$$

(a) Find the probability that his next put is over 10 m

$$P(X > 1) = \int_1^2 \frac{3}{4}x(x-2)^2\,dx$$

$$= \int_1^2 \frac{3}{4}x\,(x^2 - 4x + 4)\,dx$$

$$= \frac{3}{4}\int_1^2 x^3 - 4x^2 + 4x\,dx$$

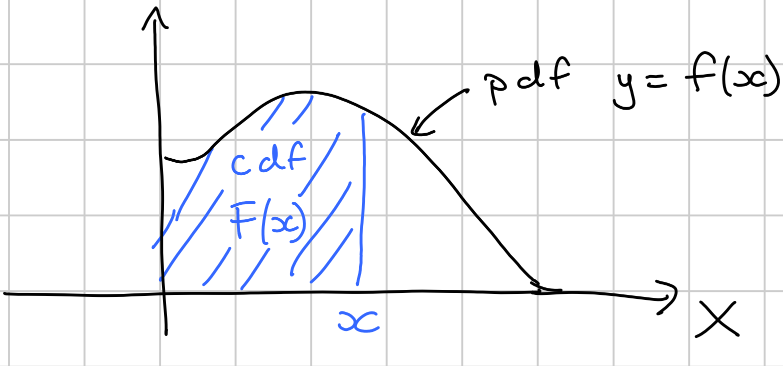
$$= \frac{3}{4}\left[\frac{1}{4}x^4 - \frac{4}{3}x^3 + 2x^2\right]_1^2$$

$$= \frac{3}{4}\left[\left(4 - \frac{32}{3} + 8\right) - \left(\frac{1}{4} - \frac{4}{3} + 2\right)\right]$$

$$= \underline{\underline{\frac{5}{16}}}$$

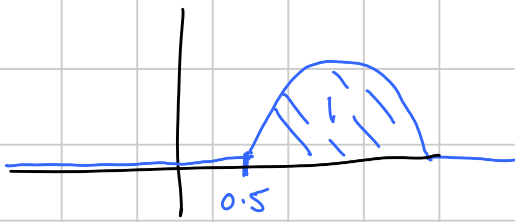
The Cumulative Distribution Function (c.d.f)

Because we often need to find areas by integrating the pdf, we define a function called the c.d.f and written $F(x)$ which gives the area under the pdf to the left of a line drawn at x



So
$$\boxed{F(x) = \int_{-\infty}^{x} f(x)\,dx}$$

in practice this is from where the area starts



Because we integrate $f(x)$ to get $F(x)$, if we are given $F(x)$ we can find $f(x)$ by differentiating: –

$$\boxed{f(x) = \frac{d}{dx}\left(F(x)\right) \quad \text{or} \quad f(x) = F'(x)}$$

Example 3 (continued)

(b) Find the cdf of X

$$F(x) = \int_0^x \frac{3}{4}x(x-2)^2\,dx$$

$$= \frac{3}{4}\left[\frac{1}{4}x^4 - \frac{4}{3}x^3 + 2x^2\right]_0^x$$

$$= \frac{3}{4}\left[\left(\frac{1}{4}x^4 - \frac{4}{3}x^3 + 2x^2\right) - (0 - 0 + 0)\right]$$

(Here we just let $x = x$!)

(This is often 0 but not always! Don't forget to do it)

$$= \frac{3}{4}\left(\frac{1}{4}x^4 - \frac{4}{3}x^3 + 2x^2\right)$$

$$= \frac{3}{16}x^4 - x^3 + \frac{3}{2}x^2$$

(c) Use the cdf to find the probability of a put

(i) over 10m

(ii) between 5 and 15m

(i) $$P(X > 1) = 1 - F(1)$$

$$= 1 - \left(\frac{3}{16} - 1 + \frac{3}{2}\right)$$

$$= \frac{5}{16}$$

(ii) $$P(0.5 < X < 1.5) = F(1.5) - F(0.5)$$

$$= \left[\frac{3}{16}(1.5)^4 - (1.5)^3 + \frac{3}{2}(1.5)^2\right] - \left[\frac{3}{16}(0.5)^4 - (0.5)^3 + \frac{3}{2}(0.5)^2\right]$$

$$= 0.6875$$

p 42 Ex 2A Q 8(b)

p 45 Ex 2B Q 1, 2, 3, 4

(b) (continued)

So $$F(x) = \frac{3}{16}x^4 - x^3 + \frac{3}{2}x^2 \qquad (0 \leqslant x \leqslant 2)$$
$$= 0 \qquad (x < 0)$$
$$= 1 \qquad (x > 2)$$

NB $F(x)$ is always 0 for any value of x to the LEFT of the active interval and 1 for any value of x to the RIGHT of the active interval

(d) Find the mean distance thrown by the shot-putter.

We know that for a discrete p.d, $E(x) = \sum xp$

For a continuous p.d, the sum becomes an integral and probabilities are replaced by the p.d.f., so

$$\boxed{E(x) = \int_{-\infty}^{\infty} x\, f(x)\, dx}$$

For the shot-putter, the pdf $f(x) = \frac{3}{4}(x^3 - 4x^2 + 4x)$
$(0 < x < 2)$

So $$E(x) = \int_0^2 x \times \frac{3}{4}(x^3 - 4x^2 + 4x)\, dx$$
$$= \frac{3}{4}\int_0^2 x^4 - 4x^3 + 4x^2 \, dx$$

$$= \frac{3}{4}\left[\frac{1}{5}x^5 - x^4 + \frac{4}{3}x^3\right]_0^2$$

$$= \frac{3}{4}\left[\left(\frac{32}{5} - 16 + \frac{32}{3}\right) - (0)\right]$$

$$= \frac{4}{5}$$

Mean distance thrown $= E(10X)$

$= 10\,E(X)$

$= \underline{\underline{8 \text{ metres}}}$

(e) Find the variance of the distance thrown.

For a discrete p.d, $\mathrm{Var}(X) = E(X^2) - [E(X)]^2$

$= \sum x^2 p - [E(X)]^2$

So, for a continuous p.d, $\mathrm{Var}(X) = E(X^2) - [E(X)]^2$

$$\boxed{\mathrm{Var}(X) = \int_{-\infty}^{\infty} x^2 f(x)\,dx - [E(X)]^2}$$

For the shot-putter,

$$E(X^2) = \int_0^2 x^2 \times \frac{3}{4}\left(x^3 - 4x^2 + 4x\right) dx$$

$$= \frac{3}{4}\int_0^2 x^5 - 4x^4 + 4x^3 \, dx$$

$$= \frac{3}{4}\left[\frac{1}{6}x^6 - \frac{4}{5}x^5 + x^4\right]_0^2$$

$$= \frac{3}{4}\left[\left(\frac{64}{6} - \frac{128}{5} + 16\right) - (0)\right]$$

$$= \frac{4}{5}$$

From part (d), $E(X) = \frac{4}{5}$, so

$$\mathrm{Var}(X) = E(X^2) - [E(X)]^2$$
$$= \frac{4}{5} - \left(\frac{4}{5}\right)^2$$
$$= \underline{\underline{\frac{4}{25}}}$$

Variance of distance thrown $= \mathrm{Var}(10X)$

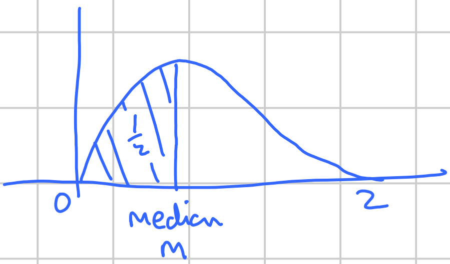
$\mathrm{Var}(aX) = a^2\,\mathrm{Var}(X)$

$$= 100\,\mathrm{Var}(X)$$
$$= 100 \times \frac{4}{25}$$
$$= \underline{\underline{16 \text{ m}^2}}$$

[Therefore the standard deviation of the distance thrown is 4 m.]

Ex 2C 1, 2, 4, 5, 6, 9

(f) Find the median distance thrown.



If the median is m, $P(X < m) = \frac{1}{2}$.

ie/ $F(m) = \frac{1}{2}$

$$\frac{3}{16}m^4 - m^3 + \frac{3}{2}m^2 = \frac{1}{2}$$

This is a quartic equation so we can only solve it by trial and improvement.

Try $m = 0.77$. LHS $= 0.4987 < \frac{1}{2}$

Try $m = 0.78$ LHS $= 0.5074 > \frac{1}{2}$

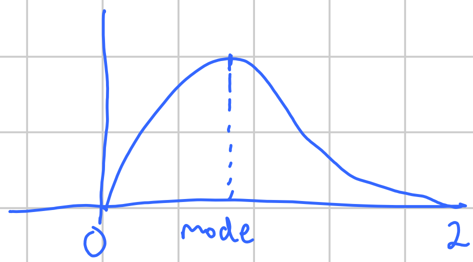
Solution is between $m = 0.77$ and $m = 0.78$, closer to 0.77

Median distance thrown $= 0.77 \times 10$

$= \underline{\underline{7.7 \text{ metres}}}$ (1dp)

NB To find Q_1, we need $F(q_1) = \frac{1}{4}$
and to find Q_3 we need $F(q_3) = \frac{3}{4}$

(g) Find the modal distance thrown



mode = value of x for which $f(x)$ is a maximum

$$f(x) = \frac{3}{4}(x^3 - 4x^2 + 4x)$$

$$f'(x) = \frac{3}{4}(3x^2 - 8x + 4)$$

We require $f'(x)$ to equal 0.

$$\frac{3}{4}(3x^2 - 8x + 4) = 0$$

$$\frac{3}{4}(3x - 2)(x - 2) = 0$$

$$x = \frac{2}{3} \quad \text{or} \quad x = 2$$

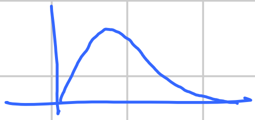
Maximum is at $x = \frac{2}{3}$

Modal distance thrown is $\frac{2}{3} \times 10 = \underline{\underline{6\frac{2}{3} \text{ metres}}}$.

Note: We don't always need to differentiate to find the mode eg

here the mode is 3

[We can see that mode < median < mean so this distribution is POSITIVELY SKEWED]



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