The Poisson Distribution

Conditions which give rise to a Poisson Distribution:

- We are interested in the occurrence of an event in a **continuum** of space or time. (The concept of a trial does not arise.)
- The random variable X is the number of times the event occurs during a **fixed interval** (eg 1 hour) of space or time
- The events occur **at random** (not at regular intervals) and **independently** (ie ~~the~~ there is no "bunching" effect).
- The **mean** number of events per interval remains **constant** (and this is the **parameter** of the distribution, called λ or sometimes μ).

Proving the formula for the probabilities of a Poisson distribution involves maths beyond C4.

Examples of situations which *may* be modelled by a Poisson distribution

1) X = The number of phone calls received by a switchboard in a 15 minute interval.

Events occur in a continuum of time
Probably random but may bunch at certain times
(Mean no per 15 mins may not be constant throughout day)

2) X = The number of dandelions in $1m^2$ of a certain field.

Events (dandelions) occur in a continuum of space.
May bunch if spores don't travel very far.

Summary of information for the Poisson Distribution:

x	p(X=x)
0	$e^{-\lambda}$
1	$\lambda e^{-\lambda}$
2	$\frac{\lambda^2}{2!}e^{-\lambda}$
3	$\frac{\lambda^3}{3!}e^{-\lambda}$
…	
r	$\frac{\lambda^r}{r!}e^{-\lambda}$
…	

$X \sim Po(\lambda)$

$E(X) = \lambda$

$Var(X) = \lambda$

(Note that the variance of a Poisson distribution is equal to the mean)

Experiment modelled by a Poisson Distribution

Note Title 18/09/2009

We looked at the results of 46 football matches played last Saturday. The random variable was:

X = the number of goals scored in a 90 minute period.

We assume goals occur at random, independently and that the mean number of goals per game is constant across the different divisions

Experimental Data

x	Frequency
0	3
1	11
2	10
3	13
4	7
5	1
6	1

$$\text{Mean } \bar{x} = \frac{\sum fx}{\sum f} = \frac{109}{46} = \underline{\underline{2.370}}$$

$$\text{Variance} = \frac{\sum fx^2}{\sum f} - \bar{x}^2 = \frac{341}{46} - \left(\frac{109}{46}\right)^2 = \underline{\underline{1.798}}$$

We observe that these calculations do not support the suggestion of a Poisson model as the mean is not even approximately equal to the variance. Either the assumptions are wrong, or the results this week are unusual.

Nevertheless we will use $X \sim Po(2.37)$ as a model, and calculate the expected frequencies.

x	$P(X=x)$	Expected Freq in 46 matches
0	$e^{-2.37} = 0.09348$	4.3
1	$2.37e^{-2.37} = 0.22155$	10.2
2	$\frac{2.37^2}{2!}e^{-2.37} = 0.26254$	12.1
3	$\frac{2.37^3}{3!}e^{-2.37} = 0.20740$	9.5
4	$= 0.12289$	5.7
5	$= 0.05825$	2.7
6	1 − above probs $= 0.03389$	1.5
	Total = 1	46.0

Comparing the expected frequencies with the actual frequencies, we see that the Poisson model is not a particularly good fit.

More examples

(1) A telephone switchboard receives on average 24 calls per hour. Assuming that a Poisson distribution is a suitable model, find:

(a) The probability that no calls are received in a given five minute interval.

Mean no of calls per five minutes $= \frac{24}{12} = 2$

Let X = no of calls in five minutes

So $X \sim Po(2)$

$$P(X=0) = e^{-2}$$
$$= \underline{0.135}$$

$$\boxed{\text{p 26 Ex 1E Q 1, 8, 9, 11}}$$

(b) the probability that less than 4 calls are received in a given 15 minute period.

Y = no of calls in 15 minutes

$Y \sim Po(6)$

$$p(Y < 4) = p(Y = 0, 1, 2, 3)$$

$$= e^{-6} + 6e^{-6} + \frac{6^2}{2!}e^{-6} + \frac{6^3}{3!}e^{-6}$$

$$= e^{-6}(1 + 6 + 18 + 36)$$

$$= 61e^{-6}$$

$$= \underline{\underline{0.151}}$$

(2) A shopkeeper finds that on average he sells 6 of a certain type of camera battery per week. He stocks up each Monday morning so that he starts the week with 9 batteries.

(a) What is the probability that he has to disappoint a customer during the week.

Let X = no of customers requesting a battery

$X \sim Po(6)$

$$p(X > 9) = 1 - p(X \leq 9)$$

$$= 1 - 0.9161$$

$$= \underline{\underline{0.0839}}$$

(b) What is the minimum stock level he should maintain to be 99% sure of not disappointing a customer

He should keep n in stock, where

$$P(X \leq n) \geq 0.99$$

From table, $P(X \leq 11) = 0.9799$

$P(X \leq 12) = 0.9912$

So he should start the week with 12 batteries.

Ex 1E Q12, 13, 14, 16, 17