Special (Named) Probability Distributions

A probability distribution is a theoretical mathematical object which we use as a model of a real life situation in order to analyse it and make predictions about it. We need to collect real-life data to compare with our model to see whether it fits the situation well. We would not be surprised if a small sample of data differs from the model (due to random variation), but if a large sample of data differed significantly from the model, we would conclude that the model was flawed and needed refining. For more detail on the process of mathematical modelling, see S1 Chapter 1.

For example, when considering a game played with a die and a coin, we constructed a probability distribution for the player's gain (G) per game:

G	p
-20	$\frac{5}{12}$
10	$\frac{5}{12}$
15	$\frac{1}{12}$
30	$\frac{1}{12}$

We calculated E(G) as $-\frac{5}{12}$ and concluded that on average the player would lose $\frac{5}{12}$ p per game.

We would not be surprise if results over a few games differed from this model.

However, if a large sample of results differed significantly from this model, we would have to conclude that we had overlooked something in constructing our model.

Some probability distributions can be used to model a wide variety of real life situations. Because they are so useful, they are given names and analysed in detail (for example, formulae are found for E(X) and Var(X)). Examples of such discrete probability distributions are the Uniform, the Binomial and the Poisson distributions (the last two are studied in S2). An example of a named continuous probability distribution is the Normal distribution.

We use the symbol ~ to indicate that a variable follows a certain distribution.

The Discrete Uniform Distribution

This has one **parameter**, n, which is the largest number the variable can take. Each value from 1 to n is equally likely. So the distribution is:

x	p
1	$\frac{1}{n}$
2	$\frac{1}{n}$
3	$\frac{1}{n}$
...	...
n	$\frac{1}{n}$

X ~ U(n)

$$E(X) = \frac{n+1}{2}$$

$$Var(X) = \frac{n^2-1}{12}$$

Examples

1) The number (N) obtained when rolling a die

$N \sim U(6)$ (unless the die is biased)

2) The number (M) of the month of a randomly chosen person's birthday (1 = Jan, 12 = Dec)

We could try using $M \sim U(12)$ as a model.
A large sample of data might suggest we should modify this model.

3) The last digit (D) of a randomly chosen telephone number

This can be $\{0, 1, ... 9\}$
So we could say $D = Y - 1$ where $Y \sim U(10)$
(Again we would need to test this model)

4) A number (X) chosen at random from the set {1, 3, 5, 7, 9}

Comparing X with $Y \sim U(5)$:

Y	1	2	3	4	5
X	1	3	5	7	9

we see that $X = 2Y - 1$

Further examples

① For the die $(X \sim U(6))$

(a) Find $E(X)$ and $Var(X)$.

We could do this by calculating $E(X) = \Sigma px$

and $Var(X) = E(X^2) - [E(X)]^2$

However, we have formulae for this:-

$$E(X) = \frac{6+1}{2} = 3\tfrac{1}{2}$$

$$Var(X) = \frac{6^2-1}{12} = \frac{35}{12} \text{ or } 2\tfrac{11}{12}.$$

(b) Calculate the probability that X lies within 1 s.d of the mean.

$$s.d = \sqrt{2\tfrac{11}{12}} = 1.7078\ldots$$

$$P(3.5 - 1.7078\ldots < X < 3.5 + 1.7078\ldots)$$

$$= P(1.79\ldots < X < 5.20\ldots)$$

$$= P(X = 2, 3, 4 \text{ or } 5)$$

$$= \frac{4}{6} = \underline{\underline{\frac{2}{3}}}$$

③ For D above, find $E(D)$ and $Var(D)$.

$D = Y - 1$ where $Y \sim U(10)$

$$E(Y) = \frac{10+1}{2} = 5.5$$

$$E(D) = E(Y-1)$$

$$= E(Y) - 1$$

$$= \underline{\underline{4.5}}$$

$$\mathrm{Var}(Y) = \frac{10^2 - 1}{12} = \frac{99}{12}$$

$$\begin{aligned}\mathrm{Var}(D) &= \mathrm{Var}(Y-1) \\ &= \mathrm{Var}(Y) \\ &= \underline{\underline{\frac{99}{12}}}\end{aligned}$$

(4) For $X = 2Y - 1$ where $Y \sim U(5)$, find $E(X)$ and $\mathrm{Var}(X)$

$$E(Y) = \frac{5+1}{2} = 3$$

$$\begin{aligned}E(X) &= E(2Y-1) \\ &= 2E(Y) - 1 \\ &= 2 \times 3 - 1 = \underline{\underline{5}}\end{aligned}$$

$$\mathrm{Var}(Y) = \frac{5^2 - 1}{12} = 2$$

$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{Var}(2Y-1) \\ &= 4\,\mathrm{Var}(Y) \\ &= 4 \times 2 = \underline{\underline{8}}\end{aligned}$$

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