Note Title 22/08/2013

Probability

(a) Possibility Spaces

A possibility space is the set of all possible equally likely outcomes which can occur in a situation. It can be shown by a diagram similar to a Venn Diagram.

An 'event' is a subset of these outcomes in which we are interested.

<u>Example</u> An ordinary die and a 4-sided die are thrown.

Possibility space:

TETRAHEDRON \ CUBE	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)

A B E

$n(E) = 24$

Find the following probabilities :-

(a) $p(\text{sum of the numbers} = 5) = \dfrac{n(A)}{n(E)} = \dfrac{4}{24} = \dfrac{1}{6}$

(b) $p(\text{difference in numbers} = 1) = \dfrac{n(B)}{n(E)} = \dfrac{7}{24}$

(c) $p(\text{sum} = 5 \text{ AND difference} = 1) = \dfrac{n(A \cap B)}{n(E)} = \dfrac{2}{24} = \dfrac{1}{12}$

(d) $p(\text{sum} = 5 \text{ OR difference} = 1) = \dfrac{n(A \cup B)}{n(E)} = \dfrac{9}{24} = \dfrac{3}{8}$

Note Title 10/09/2008

(b) The Addition Rule and Venn Diagrams

Two rules:–

① $p(A')$ means the probability that A does NOT happen.

$$\boxed{p(A') = 1 - p(A)}$$

② $\boxed{p(A \text{ OR } B) = p(A) + p(B) - p(A \text{ AND } B)}$

(This is based on the corresponding rule for sets:

$$n(A \cup B) = n(A) + n(B) - n(A \cap B))$$

[NB. Because $p(A \text{ AND } B)$ is $\dfrac{n(A \cap B)}{n(E)}$

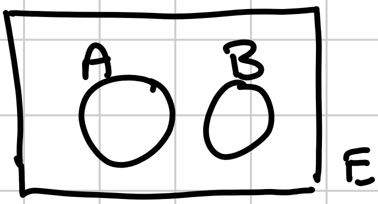
and $p(A \text{ OR } B)$ is $\dfrac{n(A \cup B)}{n(E)}$

we often write $\cap$ as shorthand for 'AND'

and $\cup$ as shorthand for 'OR']

③ Events are MUTUALLY EXCLUSIVE if $p(A \cap B) = 0$ ie they cannot both happen

In this case, $p(A \cup B) = p(A) + p(B)$



<u>Example</u> Ann takes two shots at netball

The probability that she scores with the first is $1/3$, and p(scores with second) $= \frac{1}{2}$.

The probability that she doesn't score at all is $1/4$. Find:

(a) p(she scores with the first or the second shot)

(b) p(she scores with both shots)

(c) p(she scores with the first but not the second shot)

(d) p(she scores exactly once)

Let A be the event 'she scores with first shot'
B be the event 'she scores with 2nd shot'

Then we are given

$$p(A) = 1/3 \qquad p(B) = \tfrac{1}{2} \qquad p(A' \cap B') = 1/4$$

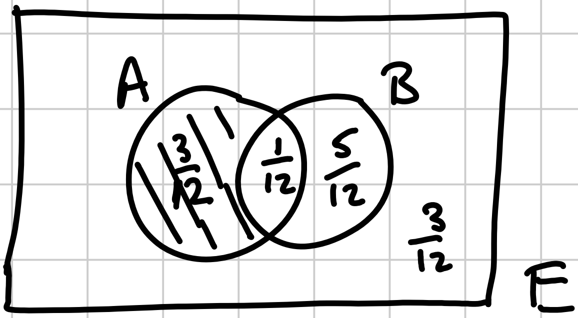
(a)
$$p(A \cup B) = 1 - p(A' \cap B')$$
$$= 1 - 1/4 = \underline{\underline{3/4}}$$

(Note that the <u>complement</u> of $A \cup B$ is $(A \cup B)'$ or $A' \cap B'$)

(b) Using

$$p(A \cup B) = p(A) + p(B) - p(A \cap B)$$
$$\frac{3}{4} = \frac{1}{3} + \frac{1}{2} - p(A \cap B)$$
$$p(A \cap B) = \frac{1}{3} + \frac{1}{2} - \frac{3}{4}$$
$$= \frac{4}{12} + \frac{6}{12} - \frac{9}{12} = \underline{\underline{\frac{1}{12}}}$$

(c)



$p(E) = 1$

$$p(A \cap B') = \frac{1}{3} - \frac{1}{12}$$
$$= \frac{4}{12} - \frac{1}{12} = \frac{3}{12} = \underline{\underline{\frac{1}{4}}}$$

(d) $p(B \cap A') = \frac{1}{2} - \frac{1}{12} = \frac{5}{12}$

p(she scores exactly once)

$$= p((A \cap B') \cup (B \cap A'))$$
$$= \frac{3}{12} + \frac{5}{12}$$
$$= \frac{8}{12} = \underline{\underline{\frac{2}{3}}}$$

p73 Ex 5A Q 1–9.

(c) Conditional Probability

A conditional probability is of the form "the probability of A <u>given that</u> B" and is written $p(A \mid B)$.

<u>Example</u>

TETRAHEDRON \ CUBE	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)

A

B

E

$n(E) = 24$

(Using the same dice as the example above)

$$p(\text{sum of dice} = 5) = \frac{4}{24} = \frac{1}{6} = \frac{n(A)}{n(E)}$$

$$p(\text{difference} = 1) = \frac{7}{24} = \frac{n(B)}{n(E)}$$

$$p(\text{sum of dice} = 5 \mid \text{difference} = 1) = \frac{n(A \cap B)}{n(B)} = \frac{2}{7}$$

$$p(\text{difference} = 1 \mid \text{sum of dice} = 5) = \frac{n(A \cap B)}{n(A)} = \frac{2}{4} = \frac{1}{2}$$

From this we see that

$$p(A \mid B) = \frac{n(A \cap B)}{n(B)}$$

By dividing top and bottom by $n(E)$ we can write this as:

$$p(A|B) = \frac{\frac{n(A\cap B)}{n(E)}}{\frac{n(B)}{n(E)}}$$

ie,
$$\boxed{p(A|B) = \frac{p(A\cap B)}{p(B)}}$$

eg above, $p(A\cap B) = \frac{2}{24}$ and $p(B) = \frac{7}{24}$

so $p(A|B) = \frac{\frac{2}{24}}{\frac{7}{24}} = \frac{2}{7}$

Similarly, $p(B|A) = \frac{p(A\cap B)}{p(A)}$

Rewriting these, we get

$$\boxed{\begin{aligned} & p(A\cap B) = p(B)\times p(A|B) \\ \text{or} \quad & p(A\cap B) = p(A)\times p(B|A) \end{aligned}}$$

<u>Examples</u>

① Two cards are drawn from a pack.

A = 'the first card is a heart'

B = 'the second card is a club'

$p(A) = \frac{13}{52} = \frac{1}{4}$

$p(B|A) = \frac{13}{51}$

so $p(A\cap B) = \frac{1}{4}\times\frac{13}{51} = \frac{13}{204}$

(2) Given that $p(B) = \frac{1}{4}$, $p(A \cap B) = \frac{1}{6}$ and $p(A|B') = 4/9$, find :-

(a) $p(A|B) = \dfrac{p(A \cap B)}{p(B)}$

$= \dfrac{\frac{1}{6}}{\frac{1}{4}} = \frac{1}{6} \times \frac{4}{1} = \underline{\underline{\frac{2}{3}}}$

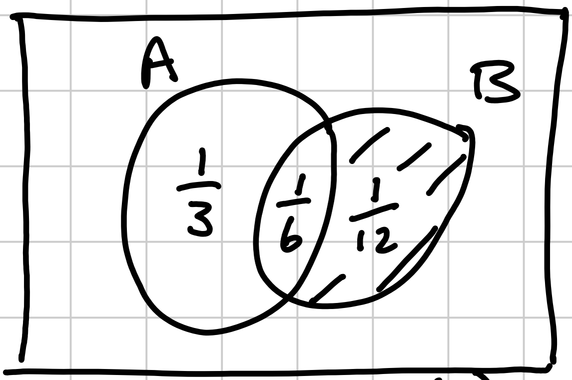
(b) $p(A \cap B') = p(B') \times p(A|B')$

$= \frac{3}{4} \times \frac{4}{9}$

$= \underline{\underline{\frac{1}{3}}}$

(c) $p(A) = \frac{1}{3} + \frac{1}{6}$

$= \underline{\underline{\frac{1}{2}}}$



$p(E) = 1$

(d) $p(B|A) = \dfrac{p(B \cap A)}{p(A)}$

$= \dfrac{1/6}{1/2} = \frac{1}{6} \times \frac{2}{1} = \underline{\underline{\frac{1}{3}}}$

(e) $p(B|A') = \dfrac{p(B \cap A')}{p(A')}$

$\left[p(B \cap A') = \frac{1}{4} - \frac{1}{6} = \frac{1}{12} \right.$ (see Venn diagram)$\left. \right]$

$$= \frac{\frac{1}{12}}{\frac{1}{2}}$$

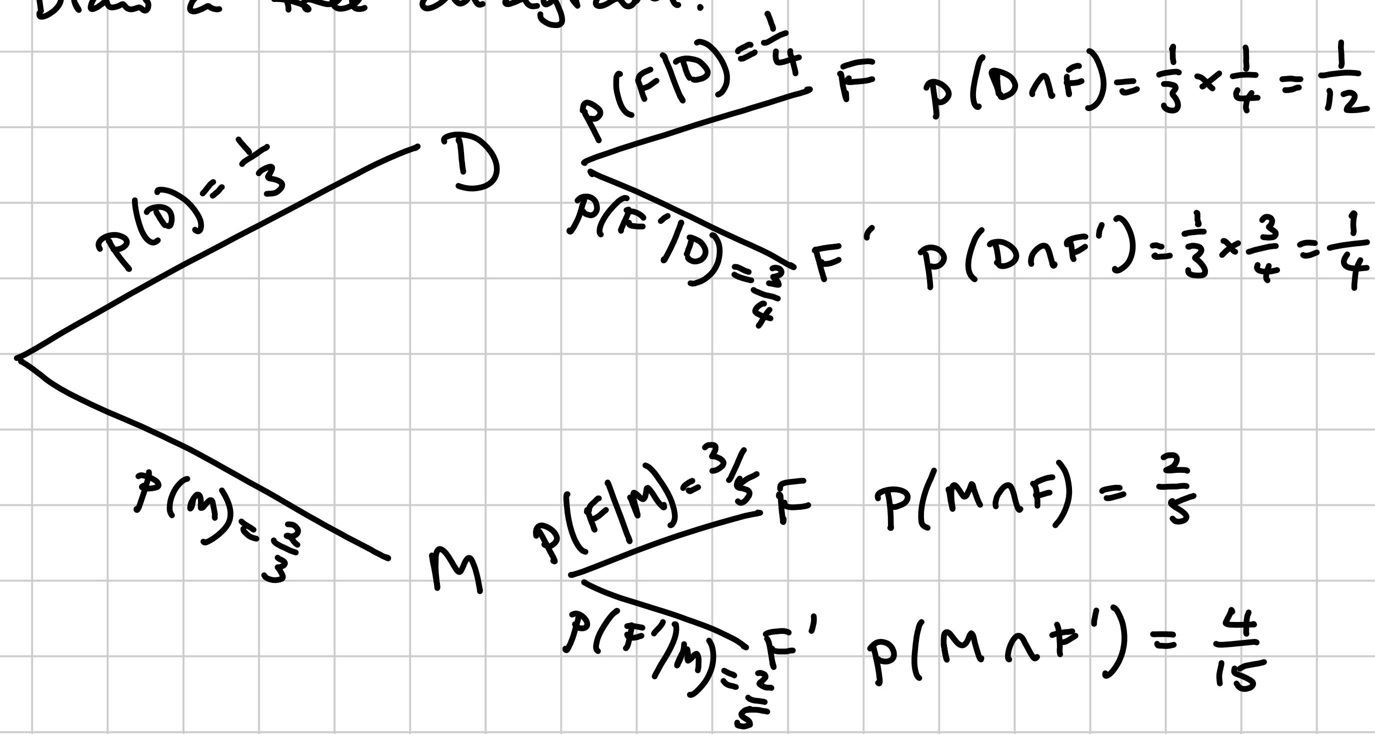
$$= \frac{1}{12} \times \frac{2}{1} = \underline{\underline{\frac{1}{6}}}$$

(d) Tree Diagrams

These are a way of showing conditional probabilites in a systematic way.

Example When I wait at the bus stop the first bus to come may be either a double decker (D) or a minibus (M), with $p(D) = \frac{1}{3}$. If it is a minibus, the probability it is full (F) is $3/5$, but if it is a D, $p(F) = 1/4$.

(a) Draw a tree diagram.



(b) What is the probability that the first bus to arrive is full?

$$p(F) = p(D \cap F) + p(M \cap F) = \frac{1}{12} + \frac{2}{5}$$

$$= \underline{\underline{\frac{29}{60}}}$$

(c) Given that the first bus is full, what is the probability that it is a double decker?

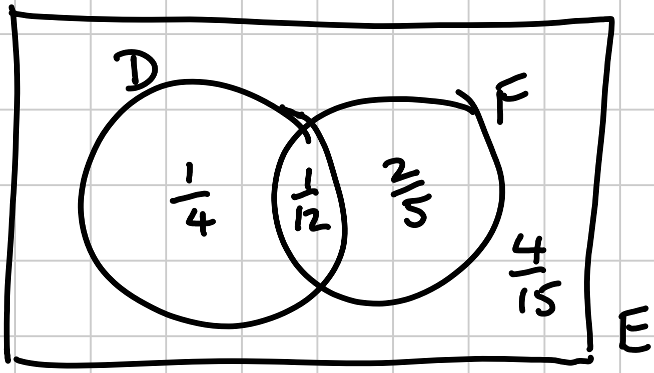
$$P(D|F) = \frac{P(D \cap F)}{P(F)} = \frac{\frac{1}{12}}{\frac{29}{60}}$$

$$= \frac{1}{12} \times \frac{60}{29}$$

$$= \underline{\underline{\frac{5}{29}}}$$

(d) Draw a Venn Diagram showing the probabilities.

(nb Minibusses $M = D'$)



(e) Given that the first bus is not full, find the probability that it is a double decker.

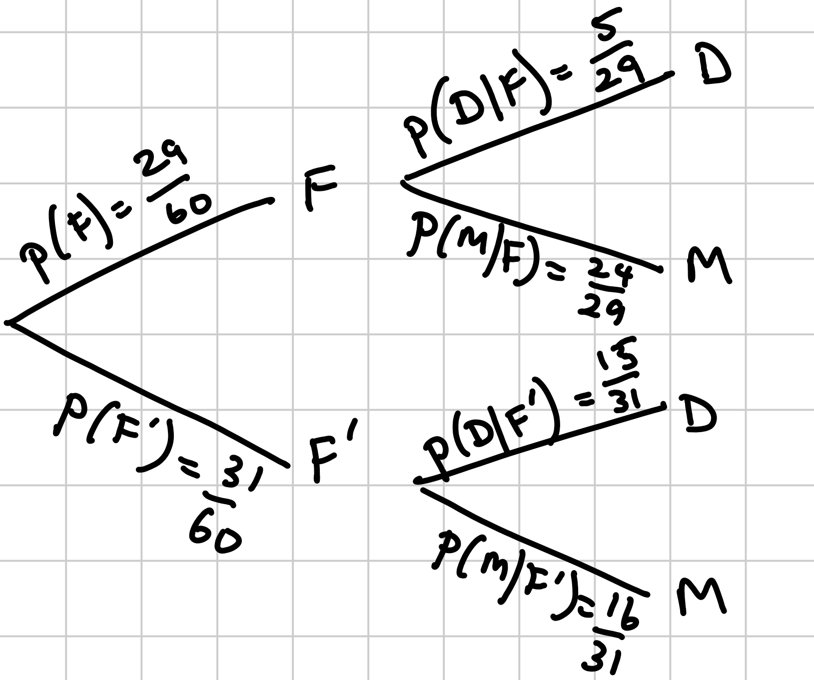
$$P(D|F') = \frac{P(D \cap F')}{P(F')}$$

$$= \frac{1/4}{31/60}$$

(from (b) $P(F') = 1 - P(F) = 1 - \frac{29}{60} = \frac{31}{60}$)

$$= \frac{1}{4} \times \frac{60}{31} = \underline{\underline{\frac{15}{31}}}$$

(f) Draw a tree diagram with F and F' on the first branches



$p(F \cap D) = \frac{29}{60} \times \frac{5}{29} = \frac{1}{12}$

$p(F \cap M) = \frac{24}{60} = \frac{2}{5}$

$p(F' \cap D) = \frac{15}{60} = \frac{1}{4}$

$p(F' \cap M) = \frac{16}{60} = \frac{4}{15}$

p78 Ex 5B Q 2, 6, 9, 10, 11, 12

(e) Independent Events

Events A and B are INDEPENDENT if the probability of A happening is not affected in any way by whether or not B occurs (and vice versa).

ie/ if

$$\boxed{P(A|B) = P(A|B') = P(A)}$$

In this case the rule

$$P(A \cap B) = P(B) \times P(A|B)$$

(which we had above) becomes

$$\boxed{p(A \cap B) = p(B) \times p(A)}$$

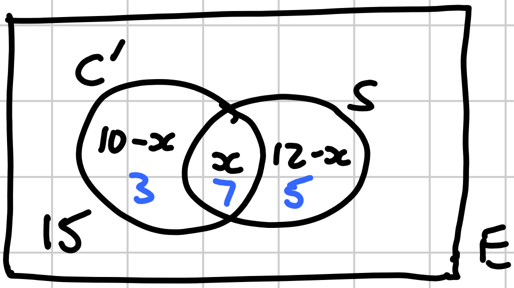
PROVIDED A AND B ARE INDEPENDENT

We can assume by common sense that events involving separate inanimate objects (eg coins, dice etc) are independent.
However, in other circumstances we cannot assume independence unless told in the question.

Examples

① Out of a class of 30 pupils, 12 are short-sighted and 10 never eat carrots. 15 of the pupils eat carrots and are not short-sighted.

(a) Draw a Venn diagram showing this data.



$$15 + 10 - x + x + 12 - x = 30$$
$$37 - x = 30$$
$$x = 7$$

(b) Find $p(C')$, $p(S)$ and $p(C' \cap S)$

$$p(C') = \frac{10}{30} = \frac{1}{3} \qquad p(S) = \frac{12}{30} = \frac{2}{5}$$

$$p(C' \cap S) = \frac{7}{30}$$

(c) Are events C' and S independent?

If they are, $p(C' \cap S) = p(C') \times p(S)$

Is $\frac{7}{30} = \frac{1}{3} \times \frac{2}{5}$?

No - so the events are <u>not</u> independent

(d) Find $p(S|C)$ and $p(S|C')$

$$= \frac{n(S \cap C)}{n(C)} \qquad = \frac{n(S \cap C')}{n(C')}$$

$$= \frac{5}{20} \qquad = \frac{7}{10}$$

$$= \frac{1}{4} \qquad = \frac{7}{10}$$

② Given that $p(A) = \frac{1}{3}$, $p(B) = \frac{1}{4}$ and A and B are independent, find $p(A \cup B)$.

Since A and B are independent,

$$p(A \cap B) = p(A) \times p(B)$$
$$= \frac{1}{3} \times \frac{1}{4}$$
$$= \frac{1}{12}$$

$$p(A \cup B) = p(A) + p(B) - p(A \cap B)$$
$$= \frac{1}{3} + \frac{1}{4} - \frac{1}{12}$$
$$= \frac{1}{2}$$

p85 Ex 5C Q 1, 4, 6, 10, 12