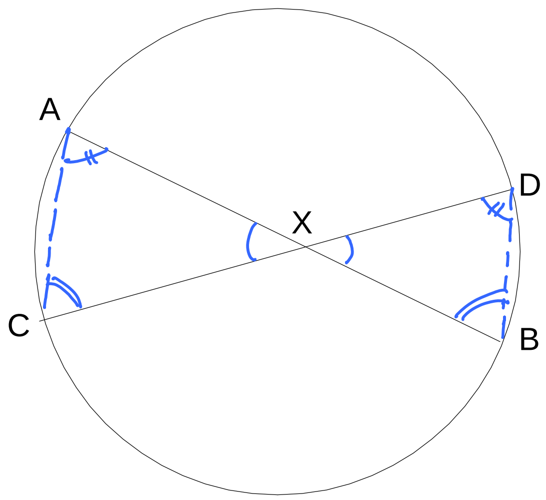
Intersecting Chords Theorem

This states that if two chords of a circle intersect as in the diagram below,

$$AX \times BX = CX \times DX$$



Proof

We can prove this by forming two triangles as shown.

$D\hat{X}B = A\hat{X}C$ (vertically opposite angles)

$A\hat{C}X = D\hat{B}X$ (angles in the same segment)

$C\hat{A}X = B\hat{D}X$ (" " " " ")

So triangles AXC and DXB are <u>similar</u>

$$\frac{AX}{DX} = \frac{CX}{BX}$$

Cross-multiply: $\boxed{AX \times BX = CX \times DX}$

Examples

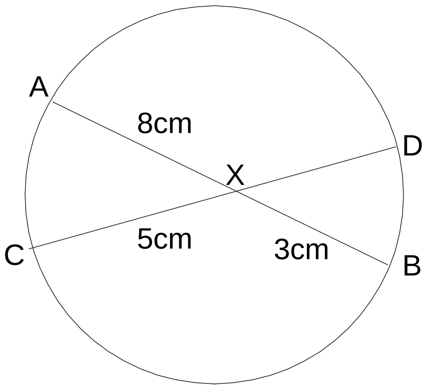
1) Find DX

$DX \times 5 = 8 \times 3$

$DX \times 5 = 24$

$DX = \frac{24}{5}$

$= \underline{\underline{4.8cm}}$



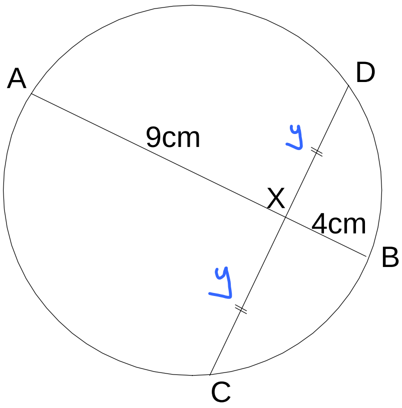
2) X is the midpoint of CD.
Find DX

Let $DX = y$

$y \times y = 9 \times 4$

$y^2 = 36$

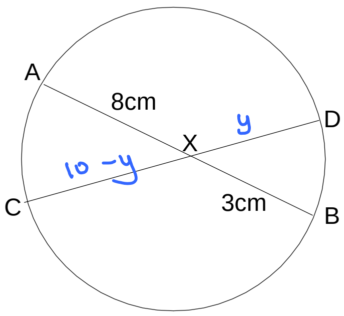
$\underline{\underline{y = 6cm}}$



3) DC = 10cm. DX is shorter than CX.
Find DX

Let $DX = y$

So $CX = 10 - y$



$y(10 - y) = 8 \times 3$

$10y - y^2 = 24$

$(-10y)\ (+y^2) \qquad (+y^2)\ (-10y)$

$0 = y^2 - 10y + 24$

$0 = (y - 4)(y - 6)$

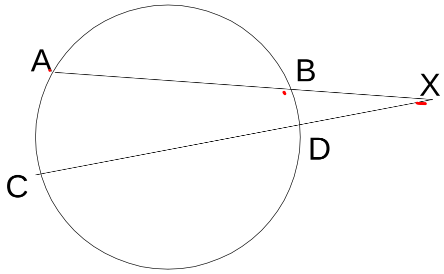
Either $y - 4 = 0$ or $y - 6 = 0$

$y = 4 \qquad y = 6$

So $\underline{DX = 4cm}$ and $CX = 6cm$

Note that the same result works if the lines cross outside the circle. But the distances to be multiplied are always from a point on the circle to the point where the lines cross:

$$AX \times BX = CX \times DX$$



$\boxed{\text{NOT } AB \times BX = CD \times DX}$

Examples

1) Find AB

First find AX

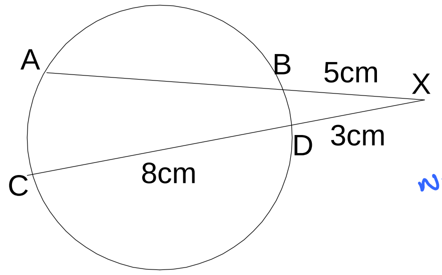
$AX \times 5 = 11 \times 3$

$AX \times 5 = 33$

$AX = \frac{33}{5}$

$= 6.6\text{cm}$

$AB = 6.6 - 5 = \underline{\underline{1.6\text{cm}}}$



NOT TO SCALE!

2) Find BX

Let $BX = y$

$(13 + y) \times y = 10 \times 3$

$13y + y^2 = 30$

$(-30) \quad (-30)$

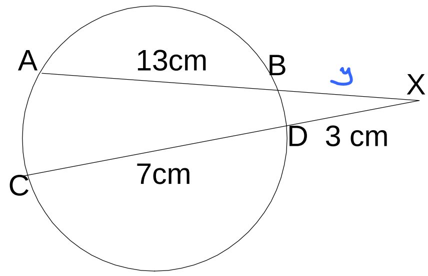
$y^2 + 13y - 30 = 0$

$(y + 15)(y - 2) = 0$

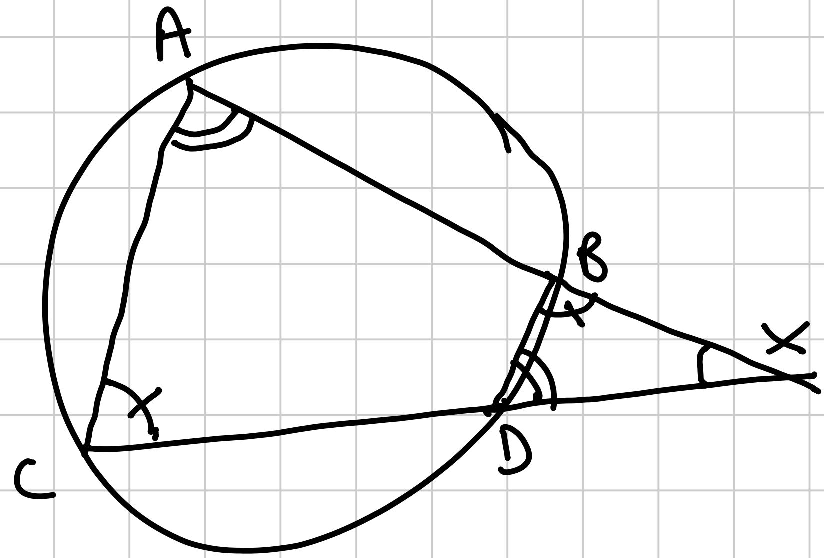
Either $y + 15 = 0$ or $y - 2 = 0$

$y = -15$ or $y = 2$

(not possible) $\underline{BX = 2\text{cm}}$



Proof of theorem when chords meet outside circle :-



$\hat{CAB} + \hat{CDB} = 180°$ (opp <s in cyclic quad)
$\hat{XDB} + \hat{CDB} = 180°$ (<s on a straight line)

$\therefore$ $\hat{CAB} = \hat{XDB}$

$\hat{ACD} + \hat{ABD} = 180°$ (opp <'s in cyclic quad)
$\hat{XBD} + \hat{ABD} = 180°$ (on straight line)

$\therefore$ $\hat{ACD} = \hat{XBD}$

$\therefore$ Triangles ACX and DBX are similar

$\therefore$ $$\frac{AX}{DX} = \frac{CX}{BX}$$

$\therefore$ $$\boxed{AX \times BX = CX \times DX}$$

which is the same theorem as above.