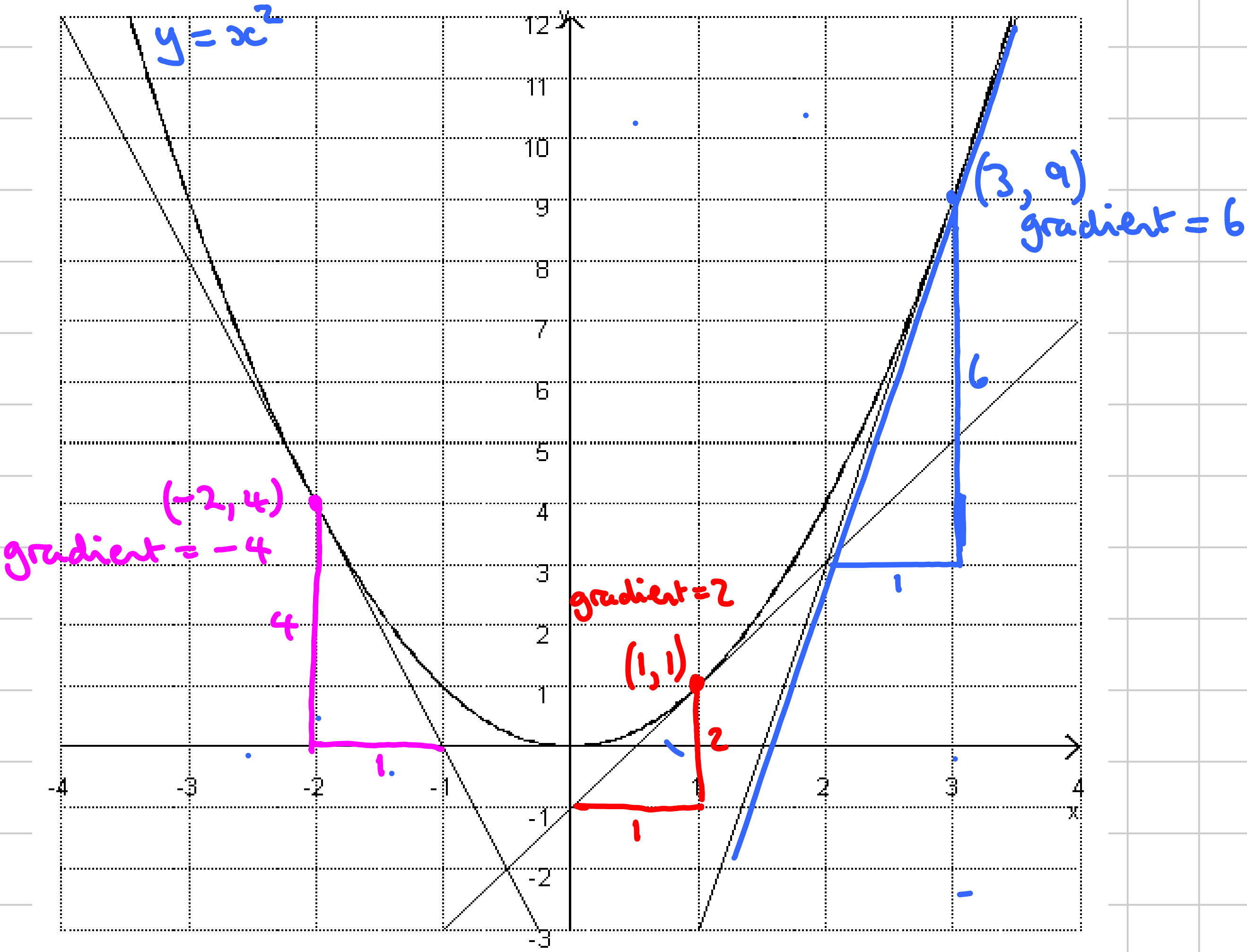
Calculus (Differentiation)

Note Title 03/11/2008

The gradient of a curve is different at each point on the curve. It is equal to the gradient of the tangent at that point. However, to find it by drawing is time-consuming and often inaccurate.

Differentiation gives us a way of finding the gradient at any point on a curve by using rules.

Illustration (see next page)



We observe that in this example, the gradient at each point is twice the x-coordinate at that point.

We write this by saying:-

if $\quad y = x^2$

then $\quad \dfrac{dy}{dx} = 2x$

[Think of "$\dfrac{dy}{dx}$" as a single symbol meaning "the formula for the gradient"]

So we can say that at the point where $x = 1.5$, the gradient is $\underline{\underline{3}}$

Each graph has a different formula for the gradient.

For example, for $y = x^3$, $\frac{dy}{dx} = 3x^2$

for $y = x^4$, $\frac{dy}{dx} = 4x^3$

The RULE is:

The gradient of the graph $y = x^n$ is given by $\frac{dy}{dx} = nx^{n-1}$

<u>Example</u> Find the gradient of the curve $y = x^4$ at the point $(2, 16)$

$$\frac{dy}{dx} = 4x^3$$

At $(2,16)$, $x = 2$, so the gradient $= 4 \times 2^3$

$= 4 \times 8$

$= \underline{\underline{32}}$

If we have a multiple of x^n, we can simply multiply $\frac{dy}{dx}$ by the same number

<u>Example</u> If $y = 5x^3$, find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = 5 \times 3x^2$$

$$= 15x^2$$

If $y =$ several terms added or subtracted, we can differentiate each term individually.

<u>Example</u> If $y = x^8 - x^6$, find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = 8x^7 - 6x^5$$

If $y = mx$, then the graph is a straight line, with gradient m at every point on the line. So $\frac{dy}{dx} = m$

<u>Example</u> If $y = -4x$, find $\frac{dy}{dx}$

$$\frac{dy}{dx} = -4$$

If $y = c$, then the graph is a horizontal line, so $\frac{dy}{dx} = 0$.

<u>Example</u> If $y = 6$, find $\frac{dy}{dx}$

$$\frac{dy}{dx} = 0$$

Putting all this together we can differentiate many different equations.

Example If $y = 2x^3 - 7x^2 + 5x - 4$, find the gradient of the graph at the point $(2, -6)$.

$y = 5x$

$y = -4$

$$\frac{dy}{dx} = 2 \times 3x^2 - 7 \times 2x + 5 + 0$$
$$= 6x^2 - 14x + 5$$

At $(2, 6)$, $x = 2$, so $\frac{dy}{dx} = 6 \times 2^2 - 14 \times 2 + 5$
$$= \underline{\underline{1}}$$

[Note that we do not need to use the y-coordinate of the point.]

Negative Indices

Remember: $2^{-3} = \frac{1}{8}$

$$x^{-2} = \frac{1}{x^2}$$

$$4x^{-1} = 4 \times \frac{1}{x} = \frac{4}{x}$$

[Only the 'x' is to the power -1; the 4 stays where it is. $(4x)^{-1}$ would be $\frac{1}{4x}$]

Examples

① Find the gradient of the curve $y = \frac{1}{x^2}$ at the point $(2, \frac{1}{4})$.

[Note: we CANNOT differentiate $\frac{1}{x^2}$ to get $\frac{1}{2x}$!]

Write $y = x^{-2}$

So $\frac{dy}{dx} = -2x^{-3}$

At $(2, \frac{1}{4})$, $x = 2$, so gradient $= -2 \times 2^{-3}$

$= -2 \times \frac{1}{8}$

$= \frac{-2}{8}$

$= -\frac{1}{4}$

② Differentiate: $y = \frac{5}{x}$

Write this as $y = 5x^{-1}$

So $\frac{dy}{dx} = 5 \times -1x^{-2}$

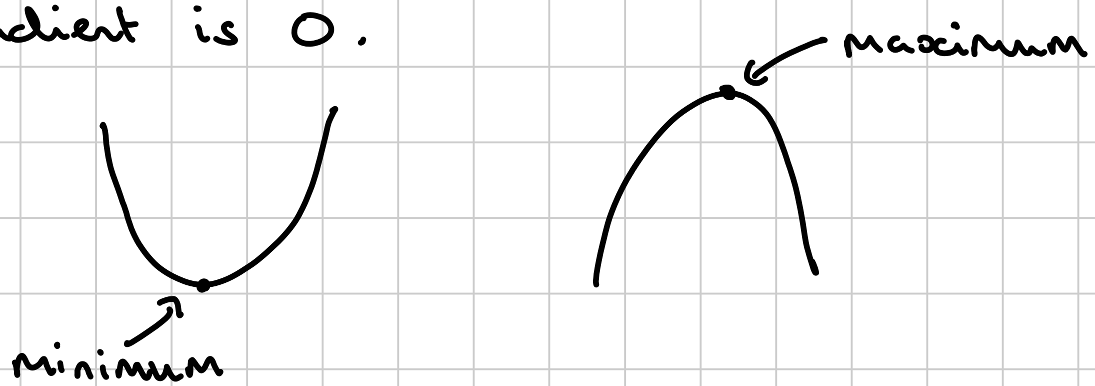
$= -5 \times \frac{1}{x^2}$

$= \frac{-5}{x^2}$

Turning Points

A turning point is a point on a graph where the gradient is 0.

e.g



Examples

① Find the turning point of the graph

of $y = x^2 + 8x + 5$. Sketch the graph showing the turning point and y-intercept

$$\frac{dy}{dx} = 2x + 8$$

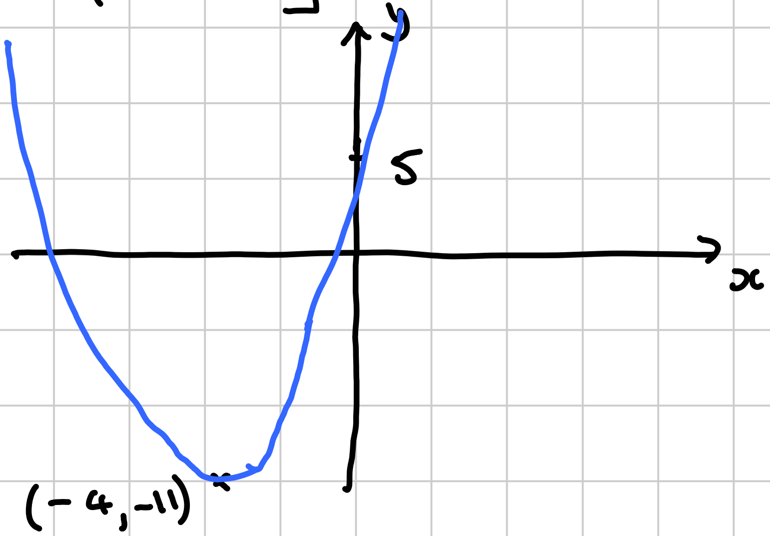
At the turning point, $2x + 8 = 0$

$(-8) \quad (-8)$

$$2x = -8$$
$$x = -4$$

So $y = (-4)^2 + 8(-4) + 5$
$= -11$

["Sketching" a graph means only plotting certain significant points.]



② Find the turning points of the graph $y = x^3 - 3x$.

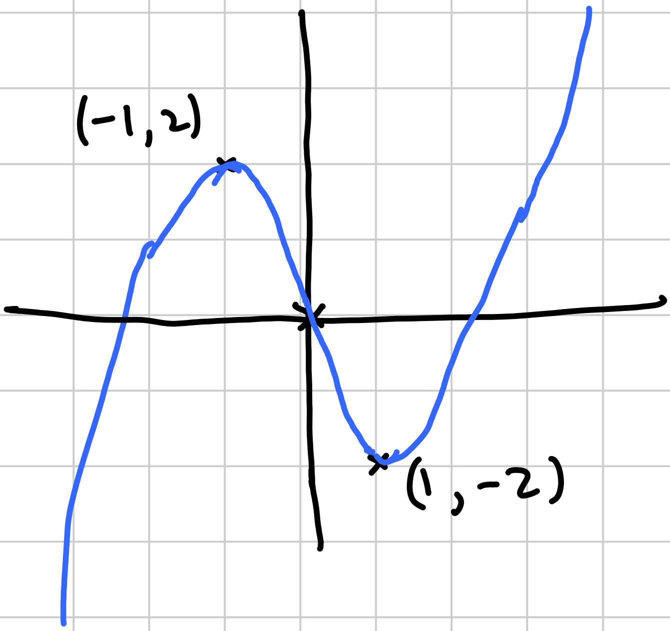
$$\frac{dy}{dx} = 3x^2 - 3$$

At turning points $3x^2 - 3 = 0$

$$3x^2 = 3$$
$$x^2 = 1$$
$$x = 1 \quad \text{or} \quad x = -1$$

If $x = 1$, $y = 1^3 - 3 \times 1 = -2$

If $x = -1$, $y = (-1)^3 - 3 \times (-1) = 2$

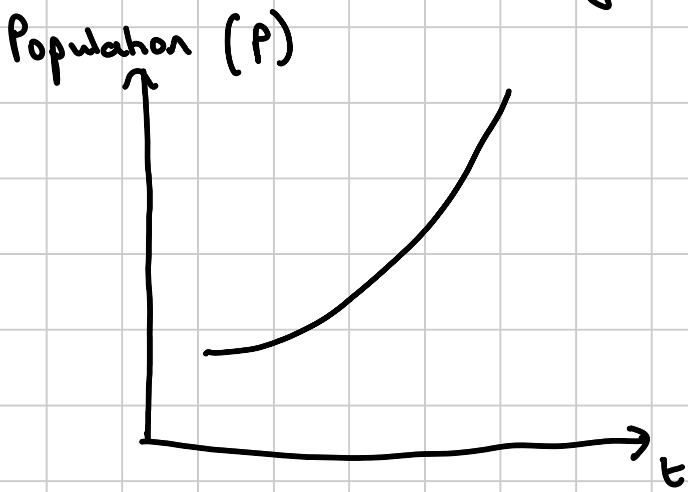


$y = x^3 - 3x + 0$

Rates of change - practical problems

The gradient of a graph gives the RATE at which the quantity on the y-axis is increasing or decreasing.

e.g.



gradient is the rate at which the population is increasing

We may have to use different letters

e.g "y" is P and "x" is t

So the gradient is given by $\frac{dP}{dt}$

Example The temperature (T) at time h hours

after midnight is given by $T = 10 + 1.4h - 0.05h^2$.
Find the rate of change of the temperature at
(a) 10 am (b) 8 pm

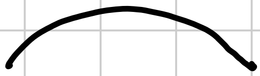
$$\frac{dT}{dh} = 1.4 - 0.05 \times 2h$$
$$= 1.4 - 0.1h$$

(a) At 10 am, $h = 10$ so $\frac{dT}{dh} = 1.4 - 0.1 \times 10$
$= \underline{0.4\ ^\circ \text{per h.}}$

(b) At 8 pm, $h = 20$ so $\frac{dT}{dh} = 1.4 - 0.1 \times 20$
$= \underline{-0.6\ ^\circ \text{per h}}$

(ie temperature is decreasing at 0.6 °C per hour)

(c) Find the time when the temperature reaches a maximum.

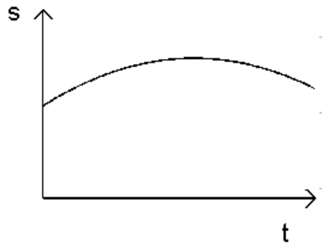


At the maximum, $\frac{dT}{dh} = 0$

$$1.4 - 0.1h = 0$$
$(+0.1h)$ $(+0.1h)$
$$1.4 = 0.1h$$
$$\frac{1.4}{0.1} = h$$
$$14 = h$$

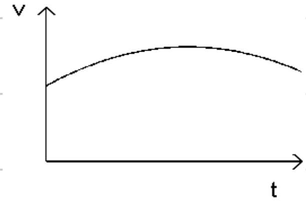
Max temperature is at 2pm

If we have a graph of distance (s) against time (t), the gradient gives the velocity.



So: $\boxed{\frac{ds}{dt} = v}$

If we have a graph of velocity (v) against time (t), the gradient gives the acceleration.



So: $\boxed{\frac{dv}{dt} = a}$

Example: A car travels between two traffic lights. The distance (s) it has travelled at time t seconds is given by the formula s = $20t - t^2$

(a) Find the formula for the velocity.
(b) Find the velocity after 5 seconds
(c) Find the formula for the acceleration.
(d) Find the acceleration after 5 seconds.

(a) $v = \frac{ds}{dt} = 20 - 2t$

(b) $v = 20 - 2 \times 5 = 10$

(c) $a = \frac{dv}{dt} = -2$

(d) $-2 \ m/s^2$

(e) $-2 \ m/s^2$

ie, the acceleration is constant.