Chapter 4 – Vectors

Note Title 28/11/2008

The Vector Product (or Cross Product)

This is defined as

$$\underline{a} \times \underline{b} = |\underline{a}|\,|\underline{b}| \sin\theta \; \hat{\underline{n}}$$

where : θ is the angle between $\underline{a}$ and $\underline{b}$

$\hat{\underline{n}}$ is a unit vector normal to both $\underline{a}$ and $\underline{b}$ (ie normal to the plane containing $\underline{a}$ and $\underline{b}$) in the direction a screw would move if turned from $\underline{a}$ to $\underline{b}$

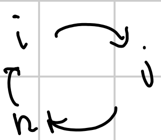
[The Scalar product (dot product) is defined as

$$\underline{a} \cdot \underline{b} = |\underline{a}|\,|\underline{b}| \cos\theta$$

and is a number, not a vector]

The definition implies that :–

1. $\underline{b} \times \underline{a} = -(\underline{a} \times \underline{b})$
2. If $\underline{a}$ and $\underline{b}$ are parallel, $\underline{a} \times \underline{b} = \underline{0}$ (the zero vector)
   In particular, $\underline{a} \times \underline{a} = \underline{0}$
3. If $|\underline{a}| = |\underline{b}| = 1$ and $\underline{a} \perp \underline{b}$, then $\underline{a} \times \underline{b} = \hat{\underline{n}}$
   (a unit vector normal to $\underline{a}$ and $\underline{b}$)
   In particular, $\underline{i} \times \underline{j} = \underline{k}$ (and $\underline{j} \times \underline{i} = -\underline{k}$)



$$\underline{j} \times \underline{k} = \underline{i}$$
$$\underline{k} \times \underline{i} = \underline{j}$$

④ $\underline{a} \times (\underline{b} + \underline{c}) = \underline{a} \times \underline{b} + \underline{a} \times \underline{c}$ (Distributive Law)

(we will assume this)

From the above we can find a formula for the product of 2 vectors in component form

Let
$$\underline{v}_1 = x_1 \underline{i} + y_1 \underline{j} + z_1 \underline{k}$$
$$\underline{v}_2 = x_2 \underline{i} + y_2 \underline{j} + z_2 \underline{k}$$

Then
$$\begin{aligned}
\underline{v}_1 \times \underline{v}_2 &= (x_1 \underline{i} + y_1 \underline{j} + z_1 \underline{k}) \times (x_2 \underline{i} + y_2 \underline{j} + z_2 \underline{k}) \\
&= \cancel{x_1 x_2 \underline{i} \times \underline{i}} + x_1 y_2 \underline{i} \times \underline{j} + x_1 z_2 \underline{i} \times \underline{k} \\
&\quad + y_1 x_2 \underline{j} \times \underline{i} + \cancel{y_1 y_2 \underline{j} \times \underline{j}} + y_1 z_2 \underline{j} \times \underline{k} \\
&\quad + z_1 x_2 \underline{k} \times \underline{i} + z_1 y_2 \underline{k} \times \underline{j} + \cancel{z_1 z_2 \underline{k} \times \underline{k}}
\end{aligned}$$

$$= (y_1 z_2 - z_1 y_2)\underline{i} + (z_1 x_2 - x_1 z_2)\underline{j} + (x_1 y_2 - y_1 x_2)\underline{k}$$

This can be remembered by writing:–

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

— components of 1st vector

— components of 2nd vector

So that the coefficient of $\underline{i}$ is found using the 'cross' shown in red above, and the same for $\underline{j}$ and $\underline{k}$ (provided we imagine the pattern to be cyclic).

<u>Example</u> Find $\begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} \times \begin{pmatrix} 4 \\ -1 \\ -5 \end{pmatrix}$

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 3 & -2 \\ 4 & -1 & -5 \end{vmatrix}\begin{matrix} \\ 1 \\ 4 \end{matrix} = -17\underline{i} - 3\underline{j} - 13\underline{k} \quad \text{or} \quad \begin{pmatrix} -17 \\ -3 \\ -13 \end{pmatrix}$$

Applications of the Vector Product

① A vector equation for a line.

We usually write a line in parametric form as

$$\underline{r} = \underline{a} + \lambda \underline{v}$$

($\underline{a}$ is the p.v. of a point on the line
$\underline{v}$ is a vector in the direction of the line
λ is a parameter - as it varies $\underline{r}$ is the p.v. of any point on the line)

We can also write this as $\boxed{(\underline{r} - \underline{a}) \times \underline{v} = \underline{0}}$

(since $\overrightarrow{AR} \parallel \underline{v}$)

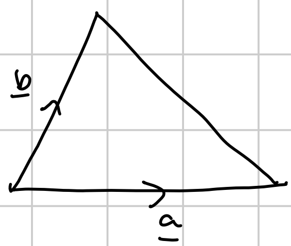
② Areas and Volumes

(a)

Area of parallelogram $= |\underline{a}||\underline{b}| \sin\theta$

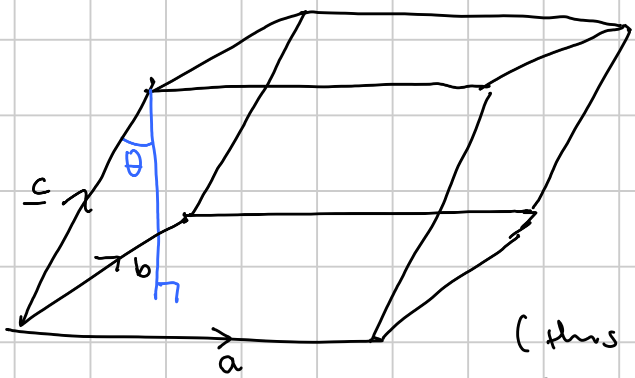
$$= |\underline{a} \times \underline{b}|$$

(b)



Area of triangle $= \frac{1}{2}|\underline{a} \times \underline{b}|$

(c)



Volume of parallelepiped

$= \text{area of base} \times \text{perp ht}$

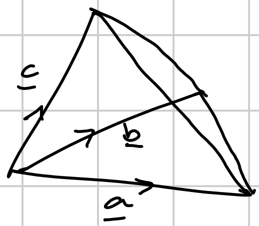
$= |a \times b|\,|c| \cos\theta$

$= |(a \times b) \cdot \underline{c}|$

(this last modulus is just to say ignore the sign if the result is negative)

[Note that $(\underline{a}\times\underline{b})\cdot c$ is usually written $a \times b \cdot c$ since this is not ambiguous because $b \cdot c$ is a number so $a \times (b \cdot c)$ doesn't make sense.]

(d) Volume of a tetrahedron



$$= \frac{1}{3}\,(\text{area of base})\ \text{height}$$
$$= \frac{1}{3}\left(\frac{1}{2}\,|\underline{a}\times\underline{b}|\right)|c|\cos\theta$$
$$= \frac{1}{6}\, a \times b \cdot c$$

<u>Example</u> Find the volume of the tetrahedron with vertices at A (1, 1, 1), B (3, 2, 1), C (2, 1, 4) and D (3, 4, 5).

$$\vec{AB} = \begin{pmatrix}2\\1\\0\end{pmatrix} \qquad \vec{AC} = \begin{pmatrix}1\\0\\3\end{pmatrix} \qquad \vec{AD} = \begin{pmatrix}2\\3\\4\end{pmatrix}$$

$$\vec{AB} \times \vec{AC} = \left|\begin{matrix}\underline{i} & \underline{j} & \underline{k}\\ 2 & 1 & 0\\ 1 & 0 & 3\end{matrix}\right|\begin{matrix}\\ 2\\ 1\end{matrix} = 3\underline{i} - 6\underline{j} - \underline{k} = \begin{pmatrix}3\\-6\\-1\end{pmatrix}$$

$$\text{So volume} = \frac{1}{6}\begin{pmatrix}3\\-6\\-1\end{pmatrix}\cdot\begin{pmatrix}2\\3\\4\end{pmatrix} = \left|\frac{1}{6}(-16)\right| = \underline{\underline{\frac{8}{3}}}$$

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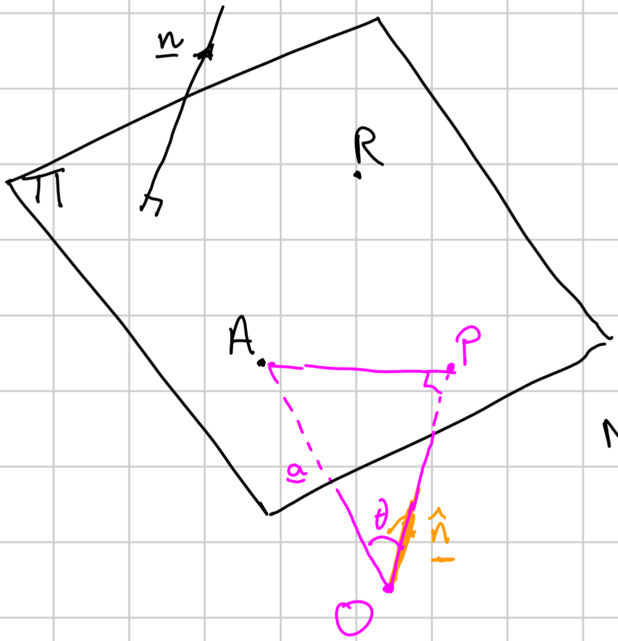
<u>Equation of a Plane</u>

The <u>direction</u> of a plane is defined by:

- 2 (non-parallel) vectors which lie in the plane

or
- 1 vector perpendicular to the plane

The <u>position</u> of a plane is fixed by the position vector of one point in the plane



Let $\hat{\underline{n}}$ be a vector normal to the plane Π, and $\underline{a}$ is the pv of A, a point on the plane.

Let $\underline{r}$ be the pv of R, a general point of the plane.

Now for any position of R

$\overrightarrow{AR} \perp \underline{n}$

$$\boxed{\begin{aligned} (\underline{r} - \underline{a}) \cdot \underline{n} &= 0 \\ \text{or} \quad \underline{r} \cdot \underline{n} &= \underline{a} \cdot \underline{n} \\ \text{or} \quad \underline{r} \cdot \underline{n} &= K \quad (\text{a constant}) \end{aligned}}$$

This is the vector equation of a plane.

Now if we make $\underline{n}$ a <u>unit</u> vector $\hat{\underline{n}}$, the equation above becomes:

$$\underline{r} \cdot \hat{\underline{n}} = \underline{a} \cdot \hat{\underline{n}}$$

But $\quad \underline{a} \cdot \hat{\underline{n}} = |\underline{a}| \cos\theta$

$\qquad\qquad = OP$ on the diagram above.

Which is the shortest distance from the origin to the plane. So in this form the eqn is

$$\boxed{\underline{r} \cdot \hat{\underline{n}} = D \quad \text{where } D \text{ is the shortest distance from the origin to the plane}}$$

If $\underline{n} = a\underline{i} + b\underline{j} + c\underline{k}$ and $\underline{r}$ is a general vector $x\underline{i} + y\underline{j} + z\underline{k}$, the equation $\underline{r} \cdot \underline{n} = K$ becomes

$$\boxed{ax + by + cz = K}$$

which is the Cartesian form of the equation of a plane

<u>Example</u> Find the vector eqn of Π, the plane normal to $\begin{pmatrix}1\\2\\-1\end{pmatrix}$ passing through $(2,-3,1)$.
Find the distance of this plane from the origin and the Cartesian eqn of the plane.

$$r \cdot \begin{pmatrix}1\\2\\-1\end{pmatrix} = \begin{pmatrix}2\\-3\\1\end{pmatrix} \cdot \begin{pmatrix}1\\2\\-1\end{pmatrix}$$

Vector eqn $\quad r \cdot \begin{pmatrix}1\\2\\-1\end{pmatrix} = -5 \qquad (r \cdot \underline{n} = K)$

$|\underline{n}| = \sqrt{1^2+2^2+(-1)^2} = \sqrt{6}$ so in the form $\underline{r} \cdot \underline{\hat{n}} = D$,

$$\underline{r} \cdot \frac{1}{\sqrt{6}}\begin{pmatrix}1\\2\\-1\end{pmatrix} = -\frac{5}{\sqrt{6}}$$

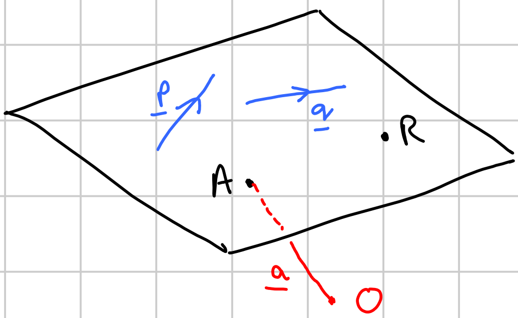
so distance from origin is $\underline{\underline{\frac{5}{\sqrt{6}}}}$ units.

Cartesian eqn : let $r = \begin{pmatrix}x\\y\\z\end{pmatrix}$

then $\begin{pmatrix}x\\y\\z\end{pmatrix} \cdot \begin{pmatrix}1\\2\\-1\end{pmatrix} = -5$

$$\Rightarrow \underline{\underline{x + 2y - z = -5}}$$

Alternatively, suppose we have 2 (non-parallel) vectors $\underline{p}$ and $\underline{q}$ which lie in the plane, and that $\underline{a}$ is still the p.v of A, a point on the plane



Then if R is a general point on the plane, $\overrightarrow{AR}$ can be written as $\lambda\underline{p} + \mu\underline{q}$ (for some λ, μ)

So $\vec{OR}$ $(= \underline{r})$ can be written as

$$\boxed{\underline{r} = \underline{a} + \lambda \underline{p} + \mu \underline{q}}$$

which is a parametric vector equation for a plane (analogous to $\underline{r} = \underline{a} + \lambda \underline{v}$ for a line).

Also $\underline{p} \times \underline{q}$ is a vector normal to the plane, so we can find a non-parametric equation for the plane

$$\boxed{\underline{r} \cdot (\underline{p} \times \underline{q}) = \underline{a} \cdot (\underline{p} \times \underline{q})}$$

<u>Example</u> Find a parametric equation and a vector equation for the plane containing the points $A(1,1,2)$ $B(2,2,1)$ and $C(1,2,3)$.

$$\vec{AB} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \vec{AC} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

so a parametric eqn could be $\underline{r} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} \begin{matrix} \\ 1 \\ 0 \end{matrix} = 2\underline{i} - \underline{j} + \underline{k}$$

So a vector equation for the plane is

$$\underline{r} \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

$$\underline{r} \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = 3$$

(and a Cartesian equation is $2x - y + z = 3$)

$\boxed{\text{p127 Ex 4C Q 1a, 2a, 6a, 7a, 10b, 11, 12}}$

Cartesian eqn of a line (in 3D)

$$\underline{r} = \underline{a} + \lambda \underline{v}$$

$$\Leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$\Leftrightarrow \left.\begin{aligned} x &= a_1 + \lambda v_1 \\ y &= a_2 + \lambda v_2 \\ z &= a_3 + \lambda v_3 \end{aligned}\right\}$$

$$\Leftrightarrow \lambda = \boxed{\frac{x - a_1}{v_1} = \frac{y - a_2}{v_2} = \frac{z - a_3}{v_3}}$$

This is the Cartesian form of the equation of a line.

Examples

① If the plane Π_1 is $\underline{r} \cdot \underline{n}_1 = c$ where $n_1 = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$, $c = 3$

and the line ℓ_1 is $r = a + \lambda \underline{p}$ where $a = \begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix}$, $p = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

find (a) the point of intersection of Π_1 and ℓ_1

(b) the angle between Π_1 and ℓ_1

Subst eqn of line into eqn of plane:—

$$\left[\begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}\right] \cdot \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} = 3$$

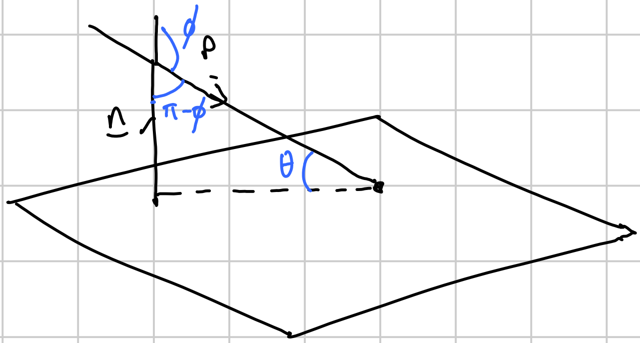
$$-8 + \lambda(-1) = 3$$

$$\lambda = -11$$

Subst into eqn of line: point of intersection is $\begin{pmatrix} 1 \\ 4 \\ 3 \end{pmatrix} + (-11)\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

ie, $(-10, 15, -19)$

(b)



$$\underline{n} \cdot \underline{p} = |\underline{n}||\underline{p}| \cos\phi$$

$$\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \sqrt{9}\sqrt{6} \cos\phi$$

$$-1 = 3\sqrt{6} \cos\phi$$

$$\cos\phi = \frac{-1}{3\sqrt{6}}$$

$$\therefore \quad \cos(\pi - \phi) = \frac{1}{3\sqrt{6}}$$

$$\therefore \quad \sin\theta = \frac{1}{3\sqrt{6}}$$

$$\underline{\underline{\theta = \arcsin\left(\frac{1}{3\sqrt{6}}\right)}}$$

(2) Given point S $(3, -2, 1)$ and the plane Π_1 above find:

(a) the foot of the perpendicular from S to Π_1

(b) the distance from S to Π_1

(c) the point S' which is the reflection of S in Π_1

(a) Eqn of line through S $\perp$ to Π_1 is

$$\underline{r} = \underline{s} + \lambda \underline{n}$$

This meets the plane where

$$(\underline{s} + \lambda \underline{n}) \cdot \underline{n} = 3$$

$$\underline{s} \cdot \underline{n} + \lambda \underline{n} \cdot \underline{n} = 3$$

$$\lambda = \frac{3 - \underline{s} \cdot \underline{n}}{\underline{n} \cdot \underline{n}}$$

$$= \frac{3 - \begin{pmatrix}3\\-2\\1\end{pmatrix}\begin{pmatrix}2\\-1\\-2\end{pmatrix}}{\begin{pmatrix}2\\-1\\-2\end{pmatrix}\begin{pmatrix}2\\-1\\-2\end{pmatrix}}$$

$$= \frac{3-6}{9} = -\frac{1}{3}$$

Subst into the line : $\underline{r} = \begin{pmatrix}3\\-2\\1\end{pmatrix} + \left(-\frac{1}{3}\right)\begin{pmatrix}2\\-1\\-2\end{pmatrix} = \begin{pmatrix}7/3\\-5/3\\5/3\end{pmatrix}$

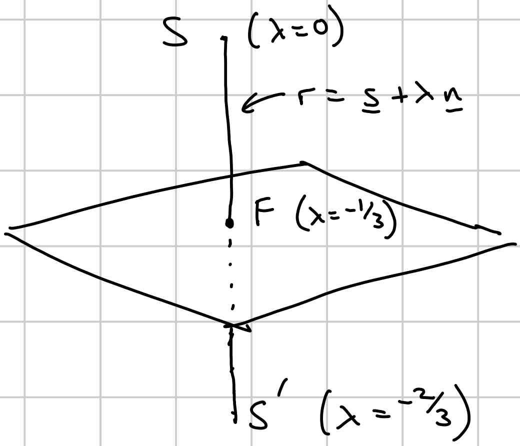
Point is $\left(7/3,\ -5/3,\ 5/3\right)$

(b) Distance from point to plane $= \left|\begin{pmatrix}3\\-2\\1\end{pmatrix} - \begin{pmatrix}7/3\\-5/3\\5/3\end{pmatrix}\right|$

$$= \left|\begin{matrix}2/3\\-1/3\\-2/3\end{matrix}\right|$$

$$= \sqrt{4/9 + 1/9 + 4/9} = \underline{\underline{1}}$$

(c)



$$\underline{s}' = \begin{pmatrix}3\\-2\\1\end{pmatrix} + (-2/3)\begin{pmatrix}2\\-1\\-2\end{pmatrix} = \begin{pmatrix}5/3\\-4/3\\7/3\end{pmatrix}$$

So S' is $\underline{\underline{\left(5/3,\ -4/3,\ 7/3\right)}}$

3 Given planes Π_1 : $r \cdot \begin{pmatrix}2\\-1\\-2\end{pmatrix} = 3$

and Π_2 : $r \cdot \begin{pmatrix}1\\2\\-2\end{pmatrix} = -1$

Find (a) their line of intersection
(b) the angle between them.

(a) This is best done using the Cartesian equations

$$\Pi_1 \qquad 2x - y - 2z = 3 \qquad ①$$
$$\Pi_2 \qquad x + 2y - 2z = -1 \qquad ②$$

- Eliminate one variable:

$$① - ② \Rightarrow \qquad x - 3y = 4$$

- Introduce a parameter:

$$\text{Let} \quad y = \lambda$$

- Express the other variables in terms of λ:

$$x = 4 + 3\lambda$$

and from ①,
$$2(4+3\lambda) - \lambda - 2z = 3$$
$$8 + 6\lambda - \lambda - 2z = 3$$
$$5 + 5\lambda = 2z$$
$$z = 5/2 + 5/2\,\lambda$$

Let $\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4+3\lambda \\ \lambda \\ 5/2 + 5/2\,\lambda \end{pmatrix}$

So $\underline{\underline{\underline{r} = \begin{pmatrix} 4 \\ 0 \\ 5/2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ 5/2 \end{pmatrix}}}$ is the line of intersection

(b) The angle between two planes is the same as the angle between their normal vectors.

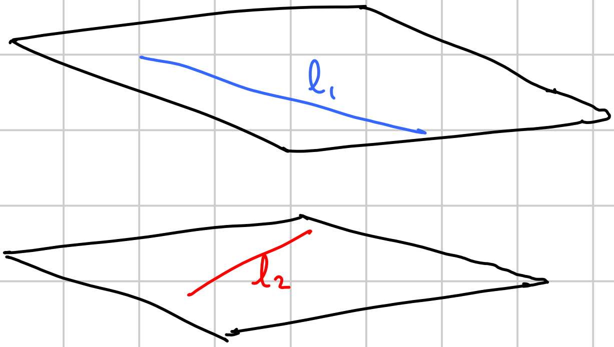
So
$$\underline{n}_1 \cdot \underline{n}_2 = |\underline{n}_1|\,|\underline{n}_2| \cos\theta$$
$$\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \sqrt{9}\,\sqrt{9} \cos\theta$$
$$4 = 9\cos\theta$$
$$\underline{\underline{\theta = \arccos\left(\tfrac{4}{9}\right)}}$$

(4) Find the shortest distance between lines

$$\ell_1 : \underline{r}_1 = \underline{a} + \lambda \underline{p} \quad \text{where } a = \begin{pmatrix}1\\4\\3\end{pmatrix}, \ p = \begin{pmatrix}1\\-1\\2\end{pmatrix}$$

$$\ell_2 : \underline{r}_2 = \underline{b} + \mu \underline{q} \quad \text{where } b = \begin{pmatrix}2\\-1\\-2\end{pmatrix} \ q = \begin{pmatrix}1\\1\\-1\end{pmatrix}$$

Let P_1 and P_2 be two parallel planes containing the lines



Then the shortest distance between the lines is equal to the distance between the two planes.

The vector $\underline{p} \times \underline{q}$ is normal to both planes

$$\text{So} \quad \underline{n} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 2 \\ 1 & 1 & -1 \end{vmatrix} = -\underline{i} + 3\underline{j} + 2\underline{k}$$

Plane containing ℓ_1 is

$$\underline{r} \cdot \begin{pmatrix}-1\\3\\2\end{pmatrix} = \begin{pmatrix}1\\4\\3\end{pmatrix} \cdot \begin{pmatrix}-1\\3\\2\end{pmatrix}$$

$$r \cdot \begin{pmatrix}-1\\3\\2\end{pmatrix} = 17$$

Make the normal a unit vector by dividing by $\sqrt{(-1)^2 + 3^2 + 2^2}$

$$r \cdot \frac{1}{\sqrt{14}} \begin{pmatrix}-1\\3\\2\end{pmatrix} = \frac{17}{\sqrt{14}}$$

Plane containing ℓ_2 is

$$r \cdot \begin{pmatrix}-1\\3\\2\end{pmatrix} = \begin{pmatrix}2\\-1\\-2\end{pmatrix} \cdot \begin{pmatrix}-1\\3\\2\end{pmatrix}$$

$$r \cdot \begin{pmatrix}-1\\3\\2\end{pmatrix} = -9$$

$$r \cdot \frac{1}{\sqrt{14}} \begin{pmatrix}-1\\3\\2\end{pmatrix} = \frac{-9}{\sqrt{14}}$$

Shortest distance between the planes (and $\therefore$ the lines)

is $\quad \dfrac{17}{\sqrt{14}} - \dfrac{-9}{\sqrt{14}} = \dfrac{26}{\sqrt{14}}$ or $\underline{\underline{\dfrac{13\sqrt{14}}{7}}}$

Algebraically: $\underline{n} = \underline{p} \times \underline{q}$ and $\hat{\underline{n}} = \frac{1}{|\underline{p} \times \underline{q}|}(\underline{p} \times \underline{q})$

Plane containing l_1 is $\underline{r} \cdot \frac{1}{|\underline{p} \times \underline{q}|}(\underline{p} \times \underline{q}) = \underline{a} \cdot \frac{1}{|\underline{p} \times \underline{q}|}(\underline{p} \times \underline{q})$

Plane containing l_2 is $\underline{r} \cdot \frac{1}{|\underline{p} \times \underline{q}|}(\underline{p} \times \underline{q}) = \underline{b} \cdot \frac{1}{|\underline{p} \times \underline{q}|}(\underline{p} \times \underline{q})$

Distance between the lines is $\frac{1}{|\underline{p} \times \underline{q}|}(\underline{p} \times \underline{q}) \cdot (\underline{a} - \underline{b})$

So a condition for two lines to intersect is

$$(\underline{p} \times \underline{q}) \cdot (\underline{a} - \underline{b}) = 0$$

(and $\underline{p} \times \underline{q} \neq \underline{0}$ otherwise the lines are parallel)