Further Vectors

Vector Product

(Remember that the scalar product $\underline{a}.\underline{b}$ is a number defined as $\underline{a}.\underline{b} = |\underline{a}||\underline{b}|\cos\theta$.)

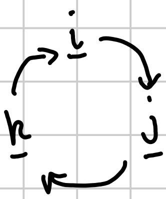
The vector product $\underline{a} \times \underline{b}$ is a vector defined as

$$\boxed{\underline{a} \times \underline{b} = |\underline{a}|\,|\underline{b}| \sin\theta\ \underline{\hat{n}}}$$

where $\underline{\hat{n}}$ is a unit vector normal to both $\underline{a}$ and $\underline{b}$ in the direction a screw would move if turned from $\underline{a}$ to $\underline{b}$.

This definition implies that :-

1. $\underline{b} \times \underline{a} = -(\underline{a} \times \underline{b})$
2. If $\underline{a}$ and $\underline{b}$ are parallel, then $\underline{a} \times \underline{b} = \underline{0}$
   In particular $\underline{a} \times \underline{a} = \underline{0}$
3. If $|\underline{a}| = |\underline{b}| = 1$, and $\underline{a}$ is perpendicular to $\underline{b}$ then $\underline{a} \times \underline{b} = \underline{\hat{n}}$
   In particular,



$\underline{i} \times \underline{j} = \underline{k}$ $\qquad$ $\underline{j} \times \underline{i} = -\underline{k}$

$\underline{j} \times \underline{k} = \underline{i}$ $\qquad$ $\underline{k} \times \underline{j} = -\underline{i}$

$\underline{k} \times \underline{i} = \underline{j}$ $\qquad$ $\underline{i} \times \underline{k} = -\underline{j}$

4. $\underline{a} \times (\underline{b} + \underline{c}) = \underline{a} \times \underline{b} + \underline{a} \times \underline{c}$ (we won't prove this)

From the above we can find a formula for the vector product of vectors in component form.

$$(x_1\underline{i} + y_1\underline{j} + z_1\underline{k}) \times (x_2\underline{i} + y_2\underline{j} + z_2\underline{k})$$

$$= \cancel{x_1 x_2\, \underline{i}\times\underline{i}} + x_1 y_2\, \underline{i}\times\underline{j} + x_1 z_2\, \underline{i}\times\underline{k}$$

$$+ y_1 x_2\, \underline{j}\times\underline{i} + \cancel{y_1 y_2\, \underline{j}\times\underline{j}} + y_1 z_2\, \underline{j}\times\underline{k}$$

$$+ z_1 x_2\, \underline{k}\times\underline{i} + z_1 y_2\, \underline{k}\times\underline{j} + \cancel{z_1 z_2\, \underline{k}\times\underline{k}}$$

$$= \underline{(y_1 z_2 - z_1 y_2)\underline{i} + (z_1 x_2 - x_1 z_2)\underline{j} + (x_1 y_2 - y_1 x_2)\underline{k}}$$

We can remember this by writing it in determinant form :—

$$\begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

To find coefficient of $\underline{i}$, find the 2×2 determinant of the numbers which remain after crossing out.

Examples

① If $\underline{a} = \underline{i} + 2\underline{j} - 3\underline{k}$ and $\underline{b} = -\underline{i} + 2\underline{k}$, find $\underline{a}\times\underline{b}$

$$\begin{vmatrix} i & j & k \\ 1 & 2 & -3 \\ -1 & 0 & 2 \end{vmatrix} = 4\underline{i} + \underline{j} + 2\underline{k}$$

② Simplify $(\underline{a}+\underline{b})\times(\underline{a}-\underline{b})$

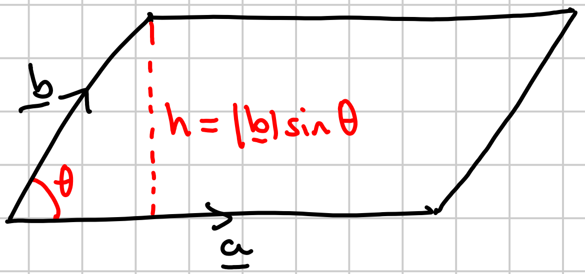
$$= \underline{a}\times\underline{a} - \underline{a}\times\underline{b} + \underline{b}\times\underline{a} - \underline{b}\times\underline{b}$$

$$= \underline{0} + \underline{b}\times\underline{a} + \underline{b}\times\underline{a} - \underline{0}$$

$$= 2(\underline{b}\times\underline{a})$$

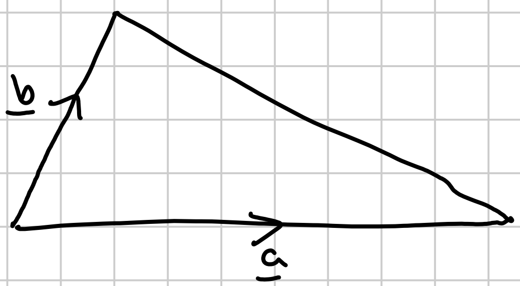
Areas and Volumes

① Parallelogram



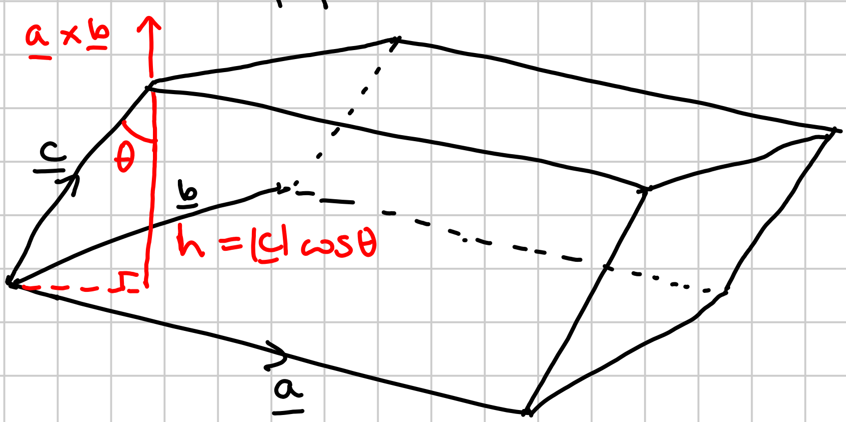
$$\text{Area} = |\underline{a}||\underline{b}| \sin\theta$$
$$= |\underline{a} \times \underline{b}|$$

② Triangle



$$\text{Area} = \tfrac{1}{2} |\underline{a} \times \underline{b}|$$

③ Parallelepiped

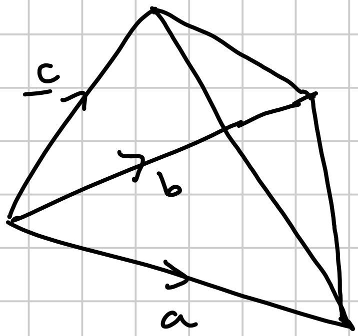


$$\text{Volume} = \text{area of base} \times \text{perp ht}$$
$$= |\underline{a} \times \underline{b}|\,|\underline{c}| \cos\theta$$
$$= (\underline{a} \times \underline{b}) \cdot \underline{c}$$

(If this is −ve, ignore the sign)

This is often written $\underline{a} \times \underline{b} \cdot \underline{c}$ since the $\times$ must be done first otherwise we can't work out $\underline{a} \times$ (scalar)

④ Volume of a tetrahedron



$$= \frac{1}{3}(\text{area of base})(\text{perp ht})$$

$$= \frac{1}{3}\left(\frac{1}{2}|\underline{a}\times\underline{b}|\right)\left(|\underline{c}|\cos\theta\right)$$

$$= \frac{1}{6}(a\times b)\cdot\underline{c}$$

p106 Ex 5A

Q 1 agh, 2, 3, 10a, 11

p110 Ex 5B

Q 4, 5(i), 8

<u>Example</u> Find the volume of the tetrahedron with vertices at $P(1,2,1)$ $Q(3,3,1)$ $R(2,2,4)$ and $S(3,5,5)$.

$$\vec{PQ} = \begin{pmatrix}2\\1\\0\end{pmatrix} = \underline{a} \quad (\underline{q} - \underline{p})$$

$$\vec{PR} = \begin{pmatrix}1\\0\\3\end{pmatrix} = \underline{b}$$

$$\vec{PS} = \begin{pmatrix}2\\3\\4\end{pmatrix} = \underline{c}$$

$$\text{Volume} = \frac{1}{6}(\underline{a}\times\underline{b}\cdot\underline{c})$$

$$\underline{a}\times\underline{b} = \begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 2 & 1 & 0\\ 1 & 0 & 3\end{vmatrix} = 3\underline{i} - 6\underline{j} - \underline{k}$$

$$\underline{a} \times \underline{b} \cdot \underline{c} = (3\times 2) - (6\times 3) - 4$$
$$= -16$$

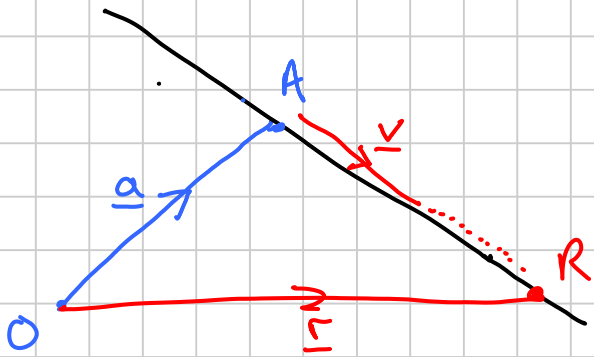
$$\text{Volume} = \left|\frac{-16}{6}\right| = 2\tfrac{2}{3} \text{ cubic units}$$

p 114 Ex 5C
Q 4, 5, 6, 10

Vector product equation of a line

The usual equation of a line is

$$\underline{r} = \underline{a} + \lambda \underline{v}$$



An alternative way of writing this is to say that

$\underline{r} - \underline{a}$ is parallel to $\underline{v}$

ie/ $$\boxed{(\underline{r} - \underline{a}) \times \underline{v} = \underline{0}}$$

Vector equation of a plane

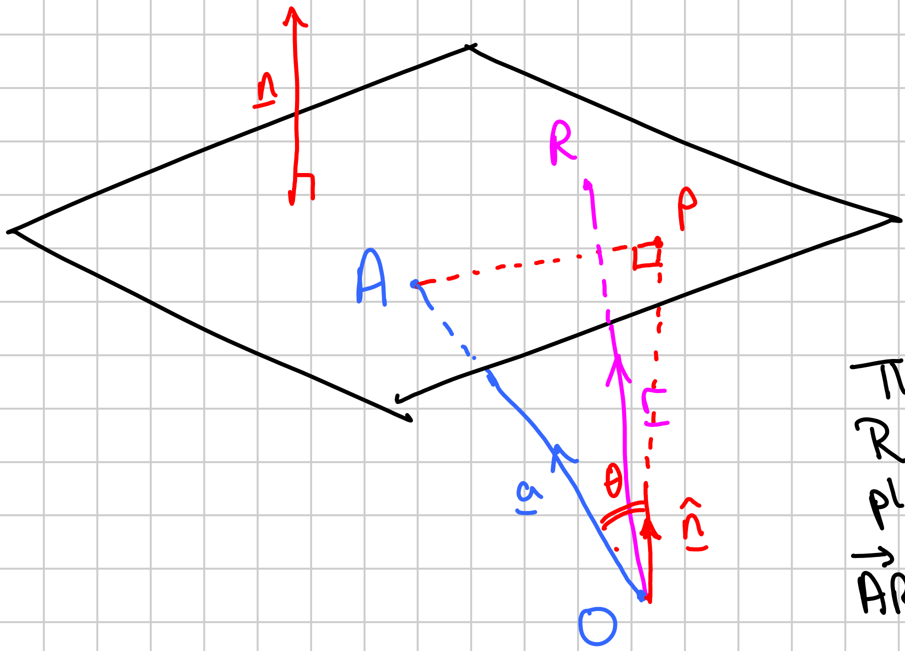
The DIRECTION of a plane can be defined by

- Two non-parallel vectors lying in the plane

OR

- ONE vector normal to the plane

The POSITION of the plane can be defined by the p.v of one point in the plane.



Let $\underline{n}$ be a normal to the plane and $\underline{a}$ be the pv of A, a point on the plane.

Then if $\underline{r}$ is the pv of R, a general point on the plane,

$\overrightarrow{AR}$ is always ⊥ to $\underline{n}$

ie, $\boxed{(\underline{r} - \underline{a}) \cdot \underline{n} = 0}$

or $\underline{r} \cdot \underline{n} - \underline{a} \cdot \underline{n} = 0$

$$\boxed{\begin{aligned} \underline{r} \cdot \underline{n} &= \underline{a} \cdot \underline{n} \\ \underline{r} \cdot \underline{n} &= K \end{aligned}}$$

← K is a constant

$\underline{n}$ could be of any length. If we make it one unit long ie, $\hat{\underline{n}}$,

$$\begin{aligned} \underline{r} \cdot \hat{\underline{n}} &= \underline{a} \cdot \hat{\underline{n}} \\ &= |a| \cos \theta \\ &= OP \quad \text{on the diagram above} \end{aligned}$$

$\boxed{\underline{r} \cdot \hat{\underline{n}} = \text{perpendicular distance from the origin to the plane.}}$

<u>Example</u> (a) Find the vector equation of the plane which is normal to $\underline{i} - 3\underline{j} + \underline{k}$ and passes through the point $(2, 0, -1)$.

$$\underline{r} \cdot (\underline{i} - 3\underline{j} + \underline{k}) = (2\underline{i} - \underline{k})(\underline{i} - 3\underline{j} + \underline{k})$$

$$\underline{r} \cdot (\underline{i} - 3\underline{j} + \underline{k}) = 1$$

(b) Hence find the distance from the origin to this plane.

Divide both sides by $|\underline{n}| = \sqrt{1^2+3^2+1^2} = \sqrt{11}$

$$\underline{r} \cdot \frac{1}{\sqrt{11}}(\underline{i} - 3\underline{j} + \underline{k}) = \frac{1}{\sqrt{11}}$$

Distance from O to plane $= \underline{\underline{\frac{1}{\sqrt{11}}}}$

Cartesian equation for plane

If we let $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$ and $\underline{n} = a\underline{i} + b\underline{j} + c\underline{k}$

then $\underline{r} \cdot \underline{n} = K$

becomes $\boxed{ax + by + cz = K.}$

which is the cartesian equation

e.g. the Cartesian equation of the plane in the example above is $\underline{\underline{x - 3y + z = 1}}$

Parametric Eqn of a Plane

Now suppose we have two vectors $\underline{p}$ and $\underline{q}$ which lie in the plane, and $\underline{a}$ is still the pv of A, a point on the plane.

Then we can form a vector $\overrightarrow{OR}$ where R can be any on the plane, by

$$\boxed{\underline{r} = \underline{a} + \lambda\underline{p} + \mu\underline{q}}$$

(this is analagous to $\underline{r} = \underline{a} + \lambda\underline{v}$ for a line)

To find a normal vector $\underline{n}$ from $\underline{p}$ and $\underline{q}$, we can work out $\underline{p} \times \underline{q}$.

Example Find a parametric equation and a vector equation for the plane containing the points A(1,1,2) B(2,2,1) and C(1,2,3)

$\underline{p} = \overrightarrow{AB} = \underline{i} + \underline{j} - \underline{k}$
$\underline{q} = \overrightarrow{AC} = \underline{j} + \underline{k}$

Parametric eqn $\quad \underline{r} = (\underline{i} + \underline{j} + 2\underline{k}) + \lambda(\underline{i} + \underline{j} - \underline{k}) + \mu(\underline{j} + \underline{k})$

$$\underline{n} = \underline{p} \times \underline{q} = \begin{vmatrix} i & j & k \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} = 2\underline{i} - \underline{j} + \underline{k}$$

Vector eqn $\quad \underline{r} \cdot (2\underline{i} - \underline{j} + \underline{k}) = (\underline{i} + \underline{j} + 2\underline{k}) \cdot (2\underline{i} - \underline{j} + \underline{k})$

$\underline{r} \cdot (2\underline{i} - \underline{j} + \underline{k}) = 3$

Cartesian eqn $\quad 2x - y + z = 3$

Ex 5E Q 1a, 2a, 3ab, 4ab, 5a, 6ab

Problems involving points, lines and planes

① Plane Π has equation $\underline{r} \cdot (2\underline{i} - \underline{j} - 2\underline{k}) = 3$

Line L has equation $\underline{r} = (\underline{i} + 4\underline{j} + 3\underline{k}) + \lambda(\underline{i} - \underline{j} + 2\underline{k})$

(a) Find the coordinates of P, the point where L meets Π.

Subst eqn of L into eqn of Π:

$$\left[(1+\lambda)\underline{i} + (4-\lambda)\underline{j} + (3+2\lambda)\underline{k}\right] \cdot (2\underline{i} - \underline{j} - 2\underline{k}) = 3$$

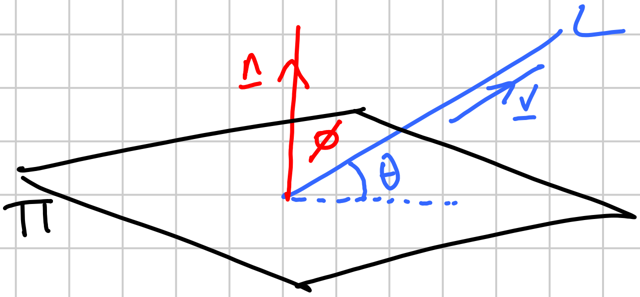
$$2(1+\lambda) - (4-\lambda) - 2(3+2\lambda) = 3$$
$$-8 - \lambda = 3$$
$$\lambda = -11$$

Subst $\lambda = -11$ into eqn of L:

$$\underline{r} = -10\underline{i} + 15\underline{j} - 19\underline{k}$$

Coordinates of P are $(-10, 15, -19)$

(b) Find the angle between Π and L.



p128 Ex 5F

Q 2(a,b), 6, 8

$$\underline{n} \cdot \underline{v} = |\underline{n}||\underline{v}| \cos\phi$$

$$(2\underline{i} - \underline{j} - 2\underline{k}) \cdot (\underline{i} - \underline{j} + 2\underline{k}) = 3\sqrt{6}\cos\phi$$

$$-1 = 3\sqrt{6}\cos\phi$$

$$\cos\phi = \frac{-1}{3\sqrt{6}}$$

$$\sin\theta = \frac{-1}{3\sqrt{6}}$$

(since $\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right)$)

$$\underline{\underline{\theta = \arcsin\left(\frac{-1}{3\sqrt{6}}\right)}}$$

2 Π_1 is the plane $\underline{r} \cdot (2\underline{i} - \underline{j} - 2\underline{k}) = 3$

Π_2 is the plane $\underline{r} \cdot (\underline{i} + 2\underline{j} - 2\underline{k}) = -1$

(a) Find the equation of the line of intersection of Π_1 and Π_2.

Write eqns in Cartesian form:

$$2x - y - 2z = 3 \qquad (1)$$
$$x + 2y - 2z = -1 \qquad (2)$$

Only 2 eqns with 3 unknowns, so can't solve completely (but we don't expect to do so).

- Eliminate one variable:

$$(1) - (2) \Rightarrow x - 3y = 4$$

- Introduce a parameter:

Let $y = \lambda$

$$\Rightarrow \quad x = 3\lambda + 4$$

- Subst back into one of the original eqns

Eqn ① $\quad 6\lambda + 8 - \lambda - 2z = 3$

$$\frac{5}{2}\lambda + \frac{5}{2} = z$$

- Eqn of line is $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$

$$\underline{r} = (3\lambda+4)\underline{i} + \lambda\underline{j} + \left(\frac{5}{2}\lambda + \frac{5}{2}\right)\underline{k}$$

or $\quad \underline{r} = \left(4\underline{i} + \frac{5}{2}\underline{k}\right) + \lambda\left(3\underline{i} + \underline{j} + \frac{5}{2}\underline{k}\right)$

(b) Find the angle between Π_1 and Π_2.

Angle between the planes = angle between normals

$$(2\underline{i} - \underline{j} - 2\underline{k}).(\underline{i} + 2\underline{j} - 2\underline{k}) = 3 \times 3 \cos\theta$$

$$4 = 9\cos\theta$$

3ab, 4

$$\theta = \arccos\left(\frac{4}{9}\right)$$

③ Π is the plane $\underline{r}.(2\underline{i} - \underline{j} - 2\underline{k}) = 3$

S is the point $(3, -2, 1)$

(a) Find the coordinates of F, the foot of the perpendicular from S to Π

The line through S normal to Π has equation

$$\underline{r} = (3\underline{i} - 2\underline{j} + \underline{k}) + \lambda(2\underline{i} - \underline{j} - 2\underline{k})$$

Subst into eqn of plane:

$$\left[(3+2\lambda)\underline{i} + (-2-\lambda)\underline{j} + (1-2\lambda)\underline{k}\right].(2\underline{i} - \underline{j} - 2\underline{k}) = 3$$

$$2(3+2\lambda) - (-2-\lambda) - 2(1-2\lambda) = 3$$

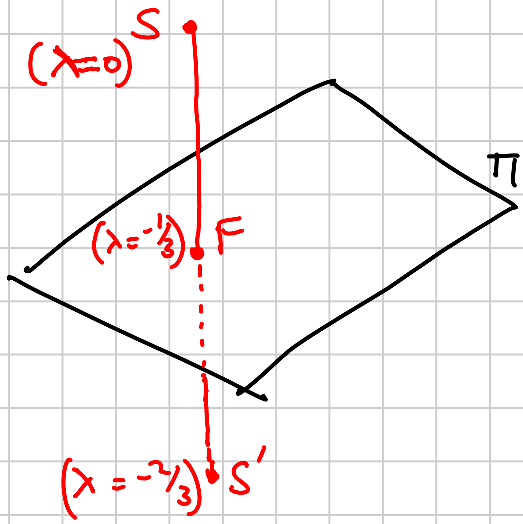
$$9\lambda + 6 = 3$$

$$\lambda = -\frac{1}{3}$$

Subst into eqn of line:

$\underline{r} = \frac{7}{3}\underline{i} - \frac{5}{3}\underline{j} + \frac{5}{3}\underline{k}$ is pv of F.

(ii) Find the coordinates of S', the reflection of S in the plane Π.



Since F is the midpoint of SS', at S' $\lambda = -\frac{2}{3}$

So pv of S' is

$\frac{5}{3}\underline{i} - \frac{4}{3}\underline{j} + \frac{7}{3}\underline{k}$

(iii) Find the shortest distance from S to Π.

<u>Method 1</u>

Find the point F as in (i).

Shortest distance is SF $= \sqrt{(\frac{7}{3}-3)^2 + (-\frac{5}{3}+2)^2 + (\frac{5}{3}-1)^2}$

$= \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{4}{9}}$

$= \underline{\underline{1}}$

<u>Method 2</u>

Let Π' be a plane through S parallel to Π.

Then Π' has eqn $\underline{r}\cdot(2\underline{i}-\underline{j}-2\underline{k}) = (3\underline{i}-2\underline{j}+\underline{k})(2i-j-2k)$

$r\cdot(2\underline{i}-\underline{j}-2\underline{k}) = 6$

Make $\underline{n}$ a unit vector $\underline{r}\cdot\frac{1}{3}(2i-j-2\underline{k}) = 2$

So distance from the origin to Π' is 2 units.

Now Π has equation $\underline{r}\cdot(2\underline{i}-\underline{j}-2\underline{k}) = 3$

$\Rightarrow \underline{r}\cdot\frac{1}{3}(2\underline{i}-\underline{j}-2\underline{k}) = 1$

So distance from the origin to Π is 1 unit.

So distance between planes $= 2-1 = \underline{\underline{1 \text{ unit}}}$

p 129 Ex 5F Q 9(a b) (parallel planes) (d) (find F)

④ Find the shortest distance between the lines

$$L_1 : \underline{r} = (\underline{i} + 4\underline{j} + 3\underline{k}) + \lambda(\underline{i} - \underline{j} + 2\underline{k})$$

$$L_2 : \underline{r} = (2\underline{i} - \underline{j} - 2\underline{k}) + \mu(\underline{i} + \underline{j} - \underline{k})$$

A vector normal to both lines is

$$\underline{n} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & -1 & 2 \\ 1 & 1 & -1 \end{vmatrix} = -\underline{i} + 3\underline{j} + 2\underline{k}$$

So a plane containing L_1 is $\underline{r}.(-i+3\underline{j}+2\underline{k}) = (i+4j+3k).(-i+3\underline{j}+2\underline{k})$

$$\underline{r}.(-\underline{i}+3\underline{j}+2\underline{k}) = 17$$

A parallel plane containing L_2 is $\underline{r}.\begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$

$$r.(-i+3j+2k) = -9$$

Making $\underline{n}$ into a unit vector: $\underline{r}.\frac{1}{\sqrt{14}}\underline{n} = \frac{17}{\sqrt{14}}$

and $r.\frac{1}{\sqrt{14}}\underline{n} = \frac{-9}{\sqrt{14}}$

Distance between lines = distance between planes

$$= \frac{17-(-9)}{\sqrt{14}} = \frac{26}{\sqrt{14}} = \underline{\underline{\frac{13\sqrt{14}}{7}}}$$

⑤ Find the distance between the lines

$$L_1 : \underline{r} = (\underline{i} + 2\underline{j} - \underline{k}) + \lambda(5\underline{i} + 4j + 3k)$$

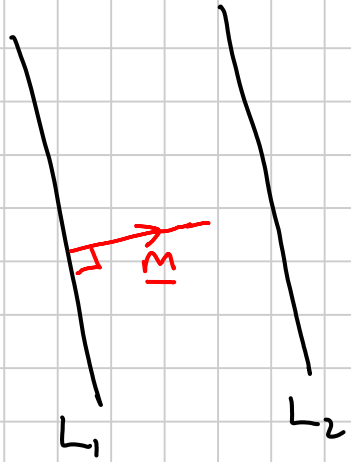
$$L_2 : \underline{r} = 2\underline{i} + \underline{k} \quad + \mu(5i + 4j + 3k)$$

The vector $(2\underline{i}+\underline{k}) - (\underline{i}+2j-\underline{k}) = \underline{i} - 2\underline{j} + 2\underline{k}$ lies in the plane containing the two lines

So a normal $\underline{n}$ to this plane is $\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 5 & 4 & 3 \\ 1 & -2 & 2 \end{vmatrix} = 14\underline{i} - 7\underline{j} - 14\underline{k}$

or $2\underline{i} - \underline{j} - 2\underline{k}$

So a normal $\underline{m}$ is $\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 5 & 4 & 3 \\ 2 & -1 & -2 \end{vmatrix} = -5\underline{i} + 16\underline{j} - 13\underline{k}$



Parallel planes containing L_1 and L_2 are

$$\underline{r}.\underline{m} = (\underline{i} + 2\underline{j} - \underline{k}).(-5\underline{i} + 16\underline{j} - 13\underline{k})$$
$$= 40$$

and

$$\underline{r}.\underline{m} = (2\underline{i} + \underline{k}).(-5\underline{i} + 16\underline{j} - 13\underline{k})$$
$$= -23$$

Distance between them is $\dfrac{40 - (-23)}{\sqrt{25 + 256 + 169}} = \dfrac{21\sqrt{2}}{10}$

[For an alternative method see p 126 in FP3 textbook.]

p 128 Ex 5F 1ab, 11, 12, 14, 15

Cartesian Eqn of a Line in 3D

The vector eqn of a line is $\underline{r} = \underline{a} + \lambda \underline{v}$

or $(\underline{r} - \underline{a}) \times \underline{v} = \underline{0}$

Let $\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\underline{a} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$ and $\underline{v} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$

Then the vector eqn becomes 3 equations:

$$x = x_1 + \lambda p$$
$$y = y_1 + \lambda q$$
$$z = z_1 + \lambda r$$

Making λ the subject of each of these:

$$\lambda = \frac{x - x_1}{p} = \frac{y - y_1}{q}$$

This is the Cartesian eqn of a line.

Examples

① Write $\frac{x-2}{3} = \frac{y+4}{1} = \frac{z-1}{2}$ in vector form

Let all these $= \lambda$

$$\Rightarrow \quad x = 3\lambda + 2$$
$$y = \lambda - 4$$
$$z = 2\lambda + 1$$

Let $\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \underline{r} = \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$

② Write $\underline{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 0 \\ 5 \end{pmatrix}$ in Cartesian form

Let $\underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow$
$$x = 1 + 4\lambda$$
$$y = 2$$
$$z = 3 + 5\lambda$$

$$(\lambda =) \ \frac{x-1}{4} = \frac{z-3}{5} \quad \text{and} \quad y = 2$$