Chapter 3 — Complex Numbers

We know that a complex number z can be written as

$$z = r(\cos\theta + i\sin\theta) \qquad (r = \text{modulus},\ \theta = \text{argument})$$

We can now find a more compact way of writing this.

Consider $e^{i\theta}$. Using the power series for e^x, this is

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \dots$$

$$= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} \dots\dots$$

$$+ i\left[\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!}\right] \dots\dots$$

$$= \cos\theta + i\sin\theta$$

So an alternative way of writing $z = r(\cos\theta + i\sin\theta)$ is $z = re^{i\theta}$

This makes it obvious that $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$: if $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$, $z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$

It also leads to de Moivre's theorem :—

$$\boxed{\begin{aligned}(\cos\theta + i\sin\theta)^n &= (e^{i\theta})^n \\ &= e^{i(n\theta)} \\ &= \cos(n\theta) + i\sin(n\theta)\end{aligned}}$$

It also gives us the links between trig and hyperbolic functions :—

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad (1)$$

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$$

$$= \cos\theta - i\sin\theta \qquad (2)$$

$$(1) + (2)) \div 2 \Rightarrow \boxed{\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \cosh(i\theta)}$$

$$((1) - (2)) \div 2 \Rightarrow \boxed{i\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2} = \sinh(i\theta)}$$

Replacing θ by $i\theta$ in these we get

$$\cos i\theta = \cosh(i^2\theta) = \cosh(-\theta) = \cosh\theta$$

and $$i\sin i\theta = \sinh(i^2\theta) = \sinh(-\theta) = -\sinh\theta$$

(multiply by i) $$-\sin i\theta = -i\sinh\theta$$

So $$\boxed{\begin{aligned}\cos(i\theta) &= \cosh\theta \\ \sin(i\theta) &= i\sinh\theta\end{aligned}}$$

This enables us to use trig identities to find hyperbolic identities, and explains Osborne's rule.

e.g ① From $\sin 2\theta = 2\sin\theta\cos\theta$

(let $\theta = ix$) $$\sin(2ix) = 2\sin(ix)\cos(ix)$$
$$i\sinh 2x = 2i\sinh x\cosh x$$
($\div\, i$) $$\sinh 2x = 2\sinh x\cosh x$$

② From $\cos^2\theta + \sin^2\theta = 1$

(let $\theta = ix$) $$\cos^2(ix) + \sin^2(ix) = 1$$
$$(\cosh x)^2 + (i\sinh x)^2 = 1$$
$$\cosh^2 x - \sinh^2 x = 1$$

We can have logs of negative numbers

e.g. to find $\ln(-2)$:

$\arg(-2) = \pi$, so write $-2 = 2e^{i\pi}$

$$\ln(-2) = \ln(2e^{i\pi})$$
$$= \ln 2 + \ln(e^{i\pi})$$
$$= \ln 2 + i\pi$$

We can solve 'impossible' trig equations

e.g $$\cos z = \frac{5}{3} \quad (>1\ !)$$

$$\Rightarrow \cosh(iz) = \frac{5}{3}$$
$$iz = \operatorname{arcosh} 5/3$$
$$= \pm\ln\left(5/3 + \sqrt{25/9 - 1}\right)$$
$$= \pm\ln 3$$
$$z = \pm i\ln 3$$

Find i^i: $|i| = 1$, $\arg(i) = \frac{\pi}{2}$ so $i = e^{i\pi/2}$. So $i^i = \left(e^{i\frac{\pi}{2}}\right)^i = e^{-\pi/2}$
$= 0.2078\ldots$

We can also see Euler's famous result linking the 5 most fundamental numbers:

$$e^{i\pi}+1 = \cos\pi + i\sin\pi + 1$$
$$= -1 + 0 + 1$$
$$\boxed{e^{i\pi}+1 = 0}$$

Using real and imaginary parts

$$\mathrm{Re}(x+iy) = x, \quad \mathrm{Im}(x+iy) = y$$

Take care:

$$\mathrm{Re}(z_1 \pm z_2) = \mathrm{Re}(z_1) \pm \mathrm{Re}(z_2)$$

BUT $\mathrm{Re}(z_1 z_2) \neq \mathrm{Re}(z_1)\,\mathrm{Re}(z_2)$

Trig Identities

We can use de Moivre's theorem to find various trig identities

① Deriving trig identities for sin, cos, tan of multiple angles.

e.g. Express $\cos 6\theta$ in terms of powers of $\cos\theta$
Hence solve the equation $32x^6 - 48x^4 + 18x^2 - 1 = 0$

$$\cos 6\theta = \mathrm{Re}(\cos 6\theta + i\sin 6\theta)$$

$$\Rightarrow \cos 6\theta = \mathrm{Re}(\cos\theta + i\sin\theta)^6$$

(write $\cos\theta = c$ and $\sin\theta = s$)

$$= \mathrm{Re}\big(c^6 + 6c^5 is + 15c^4(is)^2 + 20c^3(is)^3 + 15c^2(is)^4 + 6c(is)^5 + (is)^6\big)$$

$$= c^6 - 15c^4 s^2 + 15c^2 s^4 - s^6$$

$$= c^6 - 15c^4(1-c^2) + 15c^2(1-c^2)^2 - (1-c^2)^3$$

$$= c^6 - 15c^4 + 15c^6 + 15c^2(1-2c^2+c^4) - (1-3c^2+3c^4-c^6)$$

$$= 32c^6 - 48c^4 + 18c^2 - 1$$

So $\underline{\cos 6\theta = 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1}$

To solve $32x^6 - 48x^4 + 18x^2 - 1 = 0$

Let $x = \cos\theta \Rightarrow \cos 6\theta = 0$

$$6\theta = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \text{ or } \frac{5\pi}{2} \text{ or } \frac{7\pi}{2} \text{ or } \frac{9\pi}{2} \text{ or } \frac{11\pi}{2}$$

$$\theta = \frac{\pi}{12} \text{ or } \frac{\pi}{4} \text{ or } \frac{5\pi}{12} \text{ or } \frac{7\pi}{12} \text{ or } \frac{3\pi}{4} \text{ or } \frac{11\pi}{12}.$$

$$x = \left(\cos\frac{\pi}{12} \text{ or } \cos\frac{11\pi}{12}\right) \text{ or } \left(\frac{1}{\sqrt{2}} \text{ or } -\frac{1}{\sqrt{2}}\right) \text{ or } \left(\cos\frac{5\pi}{12} \text{ or } \cos\frac{7\pi}{12}\right)$$

$$= \pm\cos\frac{\pi}{12} \text{ or } \pm\frac{1}{\sqrt{2}} \text{ or } \pm\cos\frac{5\pi}{12}$$

② Deriving trig identities for powers of sin and cos.

We showed that

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$$

In the same way we can show that

$$\cos n\theta = \frac{e^{in\theta}+e^{-in\theta}}{2} \quad \text{and} \quad \sin n\theta = \frac{e^{in\theta}-e^{-in\theta}}{2i}$$

Example: Express $\sin^4\theta$ in terms of multiple angles and hence find $\int \sin^4\theta \, d\theta$

$$\sin^4\theta = \left[\frac{e^{i\theta}-e^{-i\theta}}{2i}\right]^4$$

$$= \frac{1}{16}\Big[e^{i4\theta} + 4e^{i3\theta}(-e^{-i\theta}) + 6e^{i2\theta}(-e^{-i\theta})^2 + 4e^{i\theta}(-e^{-i\theta})^3 + (-e^{-i\theta})^4 \Big]$$

$$= \frac{1}{16}\Big[e^{i4\theta} - 4e^{i2\theta} + 6 - 4e^{-i2\theta} + e^{-i4\theta} \Big]$$

$$= \frac{1}{16}\Big[(e^{i4\theta} + e^{-i4\theta}) - 4(e^{i2\theta} + e^{-i2\theta}) + 6 \Big]$$

$$= \frac{1}{16}\Big[2\cos 4\theta - 4(2\cos 2\theta) + 6 \Big]$$

$$= \underline{\underline{\frac{1}{8}\Big[\cos 4\theta - 4\cos 2\theta + 3 \Big]}}$$

Hence $\displaystyle\int \sin^4\theta \, d\theta = \frac{1}{8}\int \cos 4\theta - 4\cos 2\theta + 3 \, d\theta$

$$= \underline{\underline{\frac{1}{8}\Big[\frac{1}{4}\sin 4\theta - 2\sin 2\theta + 3\theta \Big] + c}}$$

$$\boxed{\text{Ex 3.4 Q 2, 5, 6, 7}}$$

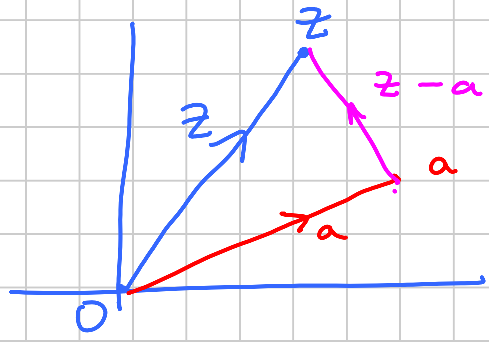
Loci in the Argand Diagram

An equation involving z is satisfied by a set of points (a locus).

We can sometimes see this geometrically, but if we can't we can fall back to writing $z = x + iy$ and finding a Cartesian equation for the locus.

Examples

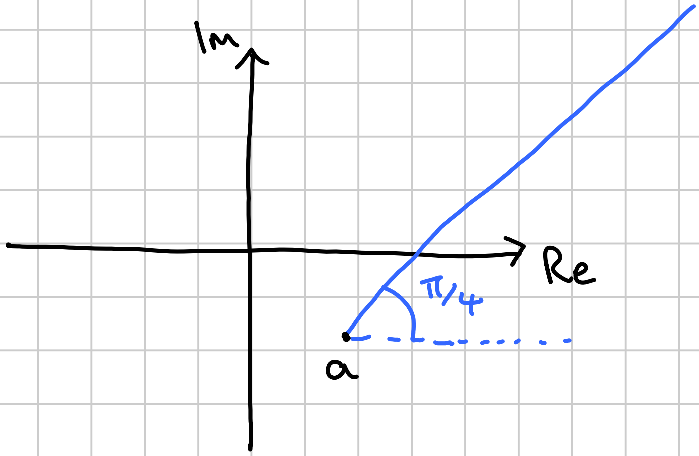
(1) $z - a$ can be represented by a vector from the point a to the point z.



So $|z - a|$ is the distance from a to z.

So the equation $|z - a| = 3$ is a circle, centre a and radius 3.

(2) $\arg(z - a) = \frac{\pi}{4}$

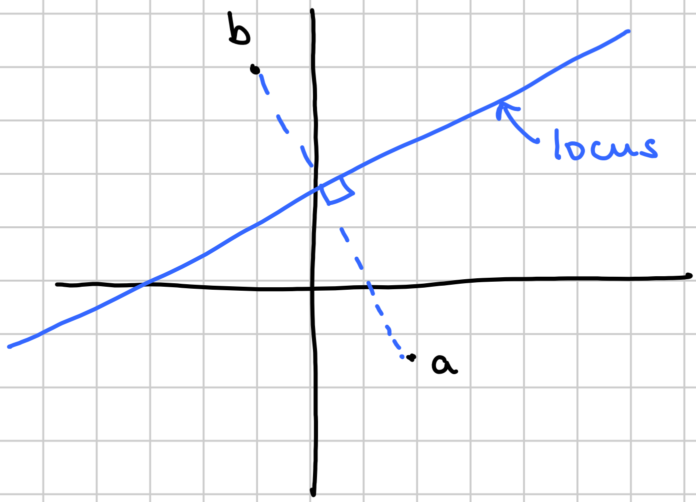


The locus is a 'half-line' or 'ray'.

(3) $\left| \dfrac{z - a}{z - b} \right| = 1$

This can be written as $\dfrac{|z - a|}{|z - b|} = 1$

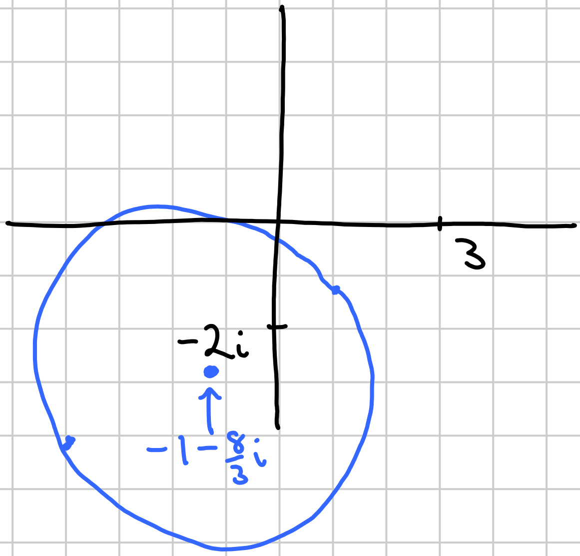
or $|z - a| = |z - b|$



locus is the perpendicular bisector of the line joining a and b

④ $\left|\frac{z-3}{z+2i}\right| = 2$ or $|z-3| = 2|z+2i|$

$= 2|z-(-2i)|$



Write this as

$$|x+iy-3| = 2|x+iy+2i|$$

$$\Rightarrow |(x-3)+iy| = 2|x+i(y+2)|$$

$$\Rightarrow \sqrt{(x-3)^2+y^2} = 2\sqrt{x^2+(y+2)^2}$$

$$\Rightarrow (x-3)^2+y^2 = 4(x^2+(y+2)^2)$$

$$\Rightarrow x^2-6x+9+y^2 = 4(x^2+y^2+4y+4)$$

$$0 = 3x^2+3y^2+6x+16y+7$$

$$0 = x^2+y^2+2x+\frac{16}{3}y+\frac{7}{3}$$

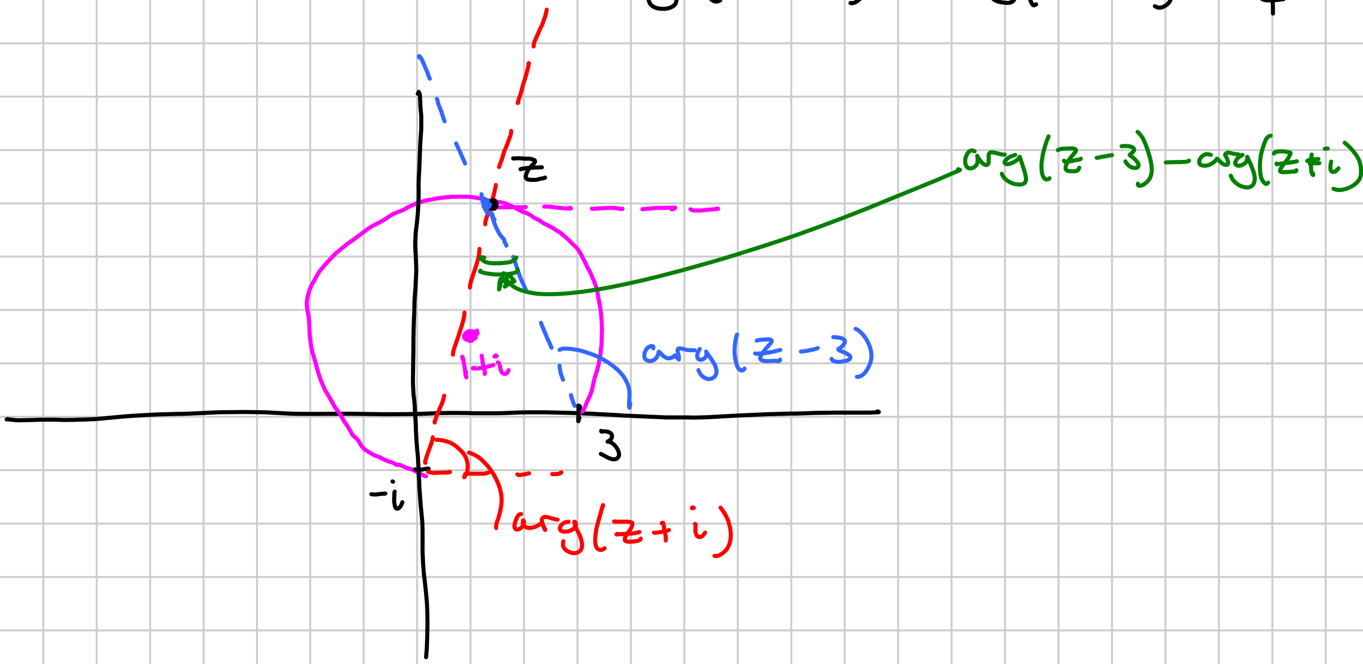
$$0 = (x+1)^2-1+\left(y+\frac{8}{3}\right)^2-\frac{64}{9}+\frac{7}{3}$$

$$\frac{52}{9} = (x+1)^2+\left(y+\frac{8}{3}\right)^2$$

which is a circle, centre $\left(-1, -\frac{8}{3}\right)$ radius $\frac{\sqrt{52}}{3}$.

(5) $\arg\left(\frac{z-3}{z+i}\right) = \frac{\pi}{4}$

Geometrically: write as $\arg(z-3) - \arg(z+i) = \frac{\pi}{4}$



By the theorem about angles in the same segment, the locus is the major arc of a circle through 3 and $-i$.

Algebraically: let $z = x + yi$

$$\arg\left(\frac{(x-3)+yi}{x+(y+1)i}\right) = \frac{\pi}{4}$$

$$\arg\left(\frac{[(x-3)+yi][x-(y+1)i]}{[x+(y+1)i][x-(y+1)i]}\right) = \frac{\pi}{4}$$

$$\arg\left(\frac{x(x-3)+y(y+1)+i[xy-(x-3)(y+1)]}{x^2+(y+1)^2}\right) = \frac{\pi}{4}$$

$$\arg\left(\frac{x^2-3x+y^2+y}{x^2+(y+1)^2} + i\,\frac{3y-x+3}{x^2+(y+1)^2}\right) = \frac{\pi}{4}$$

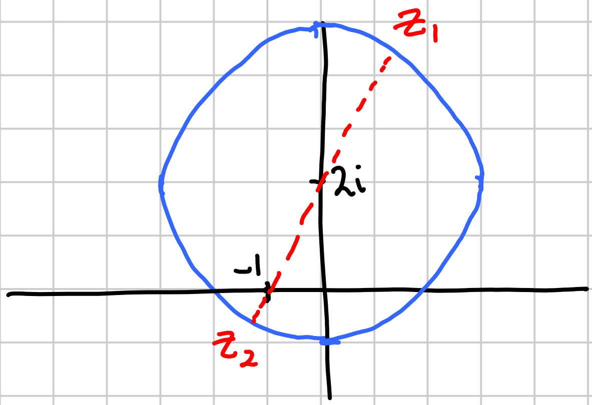
$$\frac{3y-x+3}{x^2-3x+y^2+y} = \tan\left(\frac{\pi}{4}\right) = 1$$

$$3y - x + 3 = x^2 - 3x + y^2 + y$$
$$0 = x^2 - 2x + y^2 - 2y - 3$$
$$0 = (x-1)^2 - 1 + (y-1)^2 - 1 - 3$$
$$5 = (x-1)^2 + (y-1)^2$$

Which is a circle centre $(1,1)$ or $1+i$ and radius $\sqrt{5}$.

(<u>but</u> from the geometric interpretation we only want the major arc of this circle)

(6) If $|z - 2i| = 3$ find the greatest and least values of $|z+1|$



z must lie on the blue circle.

$|z+1|$ is the distance from z to -1

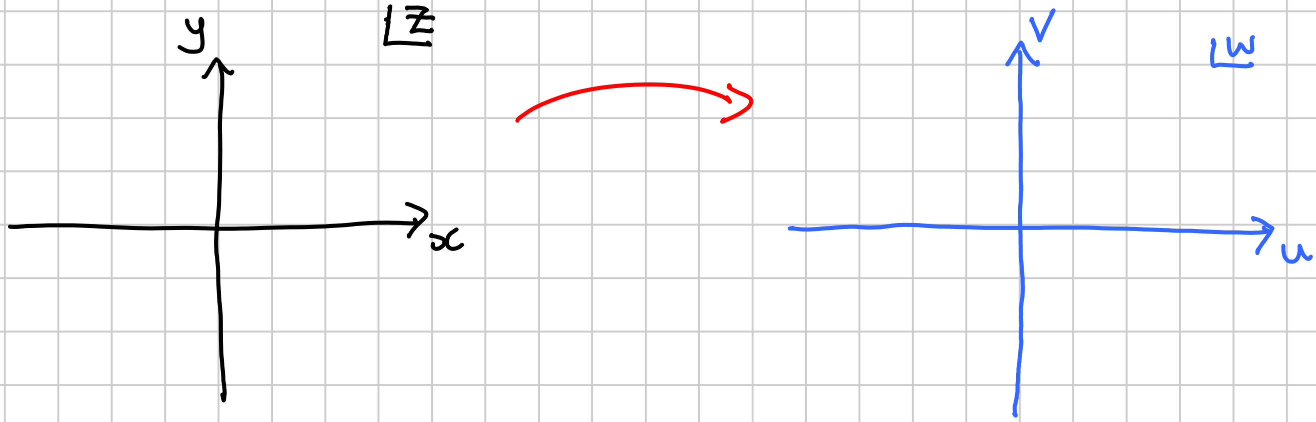
$|z+1|$ is a maximum when $z = z_1$ and is equal to
$$\sqrt{1^2+2^2} + 3 = \underline{\underline{3+\sqrt{5}}}$$

$|z+1|$ is a minimum when $z = z_2$ and is equal to $\underline{\underline{3-\sqrt{5}}}$

$\boxed{\begin{array}{l} \text{p51 Ex 3.5 Q 1bc, 2bc, 3bcd, 4, 5abc, 6abde, 7, 11, 12, 13} \\ \text{p56 Ex 3.6 Q 2ce, 3, 4, 6, 8} \end{array}}$

Transformations of the Complex Plane

If we have an equation $w = f(z)$ where z and w are both complex, we can visualize this as a transformation of the z-plane into the w-plane.



Some common transformations:

$w = z + c$	Translation $\begin{pmatrix} x \\ y \end{pmatrix}$ where $c = x + yi$
$w = -z$	Rotation π about O
$w = z^*$	Reflection in the real axis
$w = -z^*$	Reflection in the imaginary axis
$w = iz$	Rotation $\frac{\pi}{2}$ (anticlockwise)
$w = e^{i\theta} z$	Rotation θ
$w = kz$	Enlargement s.f k, centre O.

For most other transformations, it is hard to visualize the transformation as a whole. However, we can see or work out the image of a particular locus.

Principles

- Use geometric visualisation where possible
- Work with z and w where possible, but switch to x, y, u and v if necessary.

- For most equations it is useful to make z the subject first.

Examples

① Find the image of the arc $\arg\left(\frac{z}{z+i}\right) = \frac{1}{4}\pi$ under the transformation $w = \frac{z}{1-iz}$.

Change the subject

$$w(1-iz) = z$$
$$w - izw = z$$
$$w = z + izw$$
$$w = z(1+iw)$$
$$z = \frac{w}{1+iw}$$

Substitute into locus

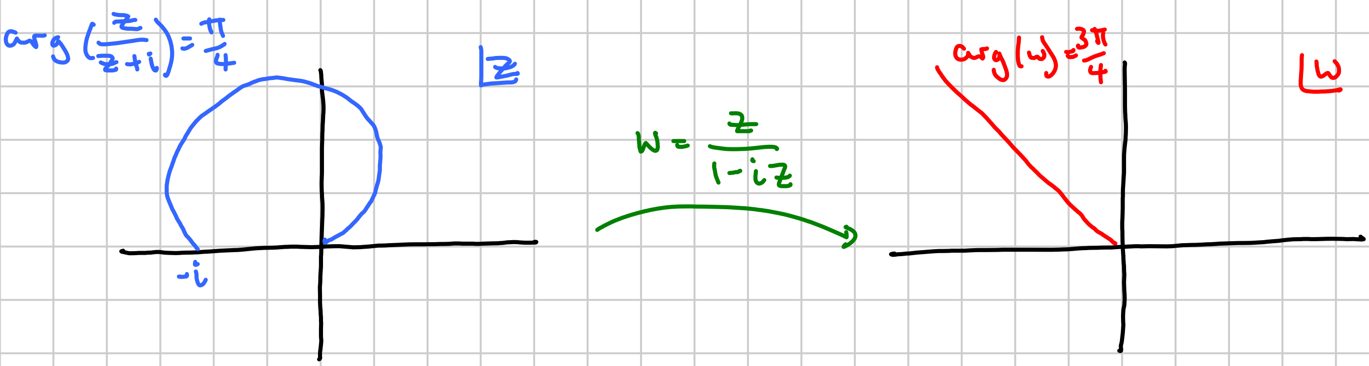
$$\arg\left(\frac{\frac{w}{1+iw}}{\frac{w}{1+iw} + i}\right) = \frac{1}{4}\pi$$

Multiply top and bottom by $1+iw$

$$\arg\left(\frac{w}{w + i(1+iw)}\right) = \frac{1}{4}\pi$$

$$\arg\left(\frac{w}{w+i-w}\right) = \frac{1}{4}\pi$$

$$\arg(w) - \arg(i) = \frac{1}{4}\pi.$$
$$\arg(w) - \frac{\pi}{2} = \frac{\pi}{4}$$
$$\arg(w) = \frac{3\pi}{4}$$



(2) Under the transformation $w = \dfrac{z - i}{z+2}$, find the image of

(a) $\text{Re}(z) = -1$ (b) $\text{Im}(z) = 0$

Make z the subject :-

$$wz + 2w = z - i$$
$$2w + i = z - wz$$
$$2w + i = z(1-w)$$
$$z = \frac{2w+i}{1-w}$$

Let $z = x + yi$ and $w = u + vi$

$$x + yi = \frac{2(u+vi)+i}{1-(u+vi)}$$
$$= \frac{2u + (2v+1)i}{(1-u) - vi} \times \frac{(1-u)+vi}{(1-u)+vi}$$
$$= \frac{2u(1-u) - v(2v+1) + i\left[2uv + (1-u)(2v+1)\right]}{(1-u)^2 + v^2}$$

$$x + yi = \frac{2u(1-u) - v(2v+1)}{(1-u)^2 + v^2} + i\left(\frac{2v + 1 - u}{(1-u)^2 + v^2}\right)$$

Now we can compare real and imaginary parts

(a) $\operatorname{Re}(z) = -1 \implies x = -1$

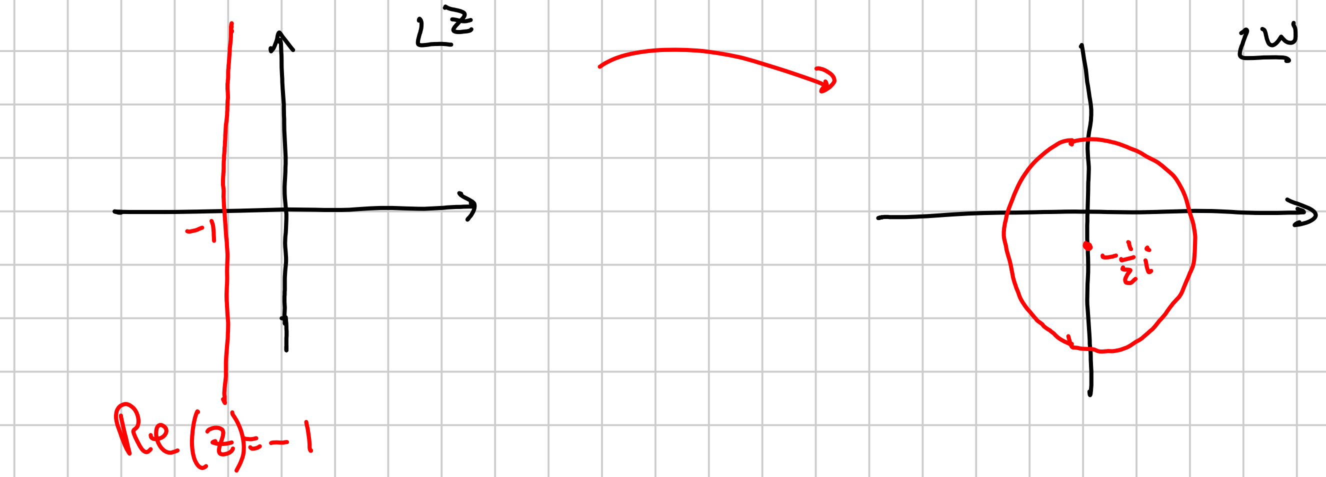
$$\implies \frac{2u - 2u^2 - 2v^2 - v}{(1-u)^2 + v^2} = -1$$

$$\implies 2u - 2u^2 - 2v^2 - v = -\left[1 - 2u + u^2 + v^2\right]$$

$$\implies 0 = u^2 + v^2 + v - 1$$

$$\implies u^2 + \left(v + \tfrac{1}{2}\right)^2 = \tfrac{5}{4}$$

Circle, centre $\left(0, -\tfrac{1}{2}\right)$ or $-\tfrac{1}{2}i$ radius $\frac{\sqrt{5}}{2}$

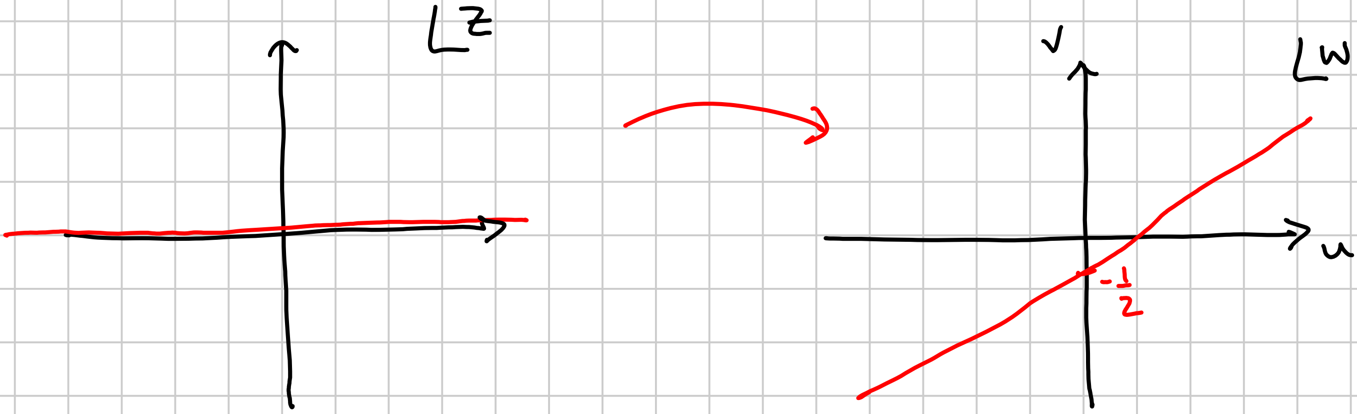


(b) $\operatorname{Im}(z) = 0 \implies y = 0$

$$\implies \frac{2v + 1 - u}{(1-u)^2 + v^2} = 0$$

$$\implies 2v + 1 - u = 0$$

$$\implies v = \tfrac{1}{2}u - \tfrac{1}{2}$$



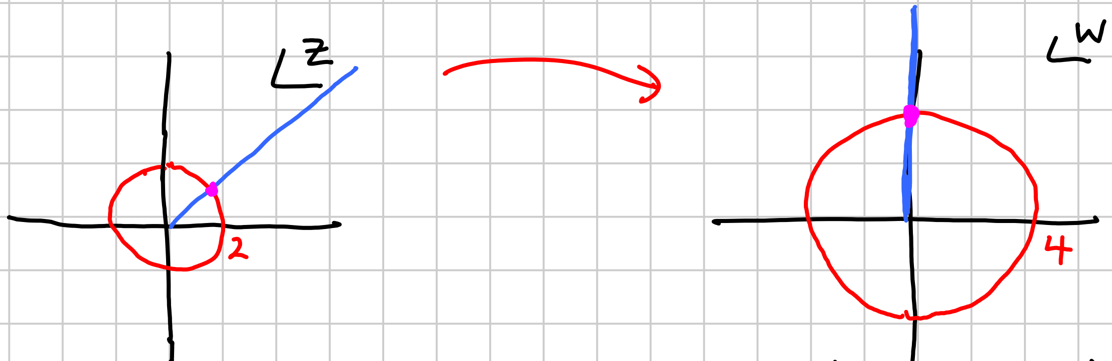
(3) Under the transformation $w = z^2$, find the image of:

(a) $|z| = 2$ (b) $\arg(z) = \frac{\pi}{4}$ (c) $\mathrm{Re}(z) + \mathrm{Im}(z) = 1$

- For this transformation, making z the subject introduces square roots which makes for complications So we avoid doing this.

(a)
$$|w| = |z^2| = |z|^2 = 4$$

$|z z| = |z||z|$



(but as the point z moves around the original circle, the point w moves around the image circle TWICE)

(b)
$$\arg(w) = \arg(z^2) = 2\arg(z) = \frac{\pi}{2}$$

(see above: image of blue line in z is the blue line in w)

(c) $\mathrm{Re}(z) + \mathrm{Im}(z) = 1$ ie, $x + y = 1$

$$w = z^2$$
$$u + vi = (x + yi)^2 = (x^2 - y^2) + 2xyi$$

So $u = x^2 - y^2 = (x+y)(x-y)$

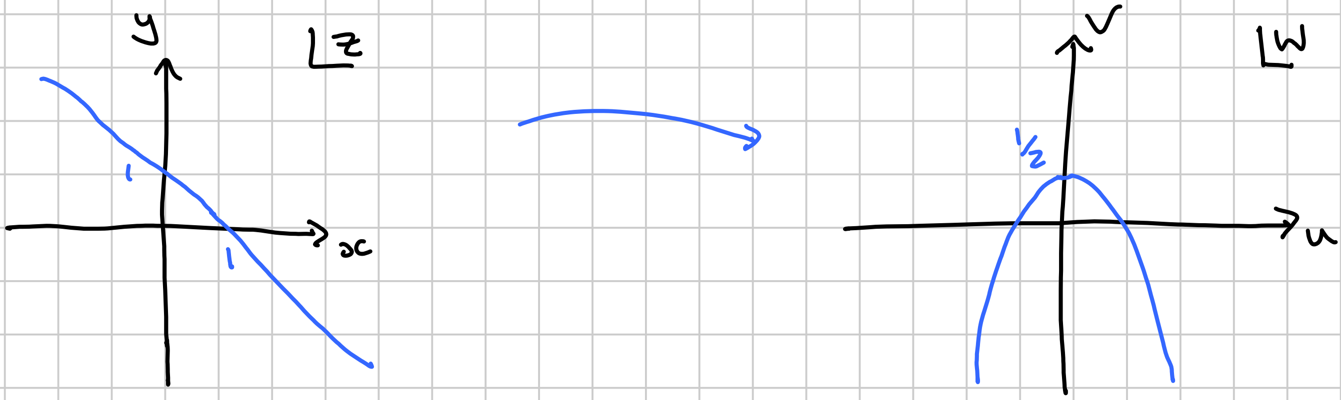
But $x+y=1 \Rightarrow u = x - y$

And $v = 2xy$

We need to eliminate x and y

$$u^2 = x^2 - 2xy + y^2$$

$$\begin{aligned} u^2 + v &= x^2 + y^2 \\ &= (x+y)^2 - 2xy \\ &= 1^2 - v \\ u^2 + 2v &= 1 \\ v &= \tfrac{1}{2} - \tfrac{1}{2}u^2 \end{aligned}$$



p63 Ex 3.7 Q 1, 2a, 4, 5, 6, 7, 8, 9b, 10

The n^{th} roots of a complex number

Any number has n n^{th} roots.

Example Find the 5^{th} roots of $w = 16 + 16\sqrt{3}i$

Write w in modulus-argument form:

$$\begin{aligned} |w| &= \sqrt{16^2 + (16\sqrt{3})^2} \\ &= \sqrt{16^2(1+3)} \\ &= 16 \times 2 = \underline{\underline{32}} \end{aligned}$$

$$\begin{aligned} \arg(w) &= \arctan\left(\frac{16\sqrt{3}}{16}\right) \\ &= \frac{\pi}{3} \end{aligned}$$

So $\quad w = 32e^{i\pi/3}$

So we need to find z such that

$$z^5 = 32e^{i\pi/3}$$

Let $\quad z = re^{i\theta} \quad \Rightarrow \quad r^5 e^{i5\theta} = 32e^{i\pi/3}$

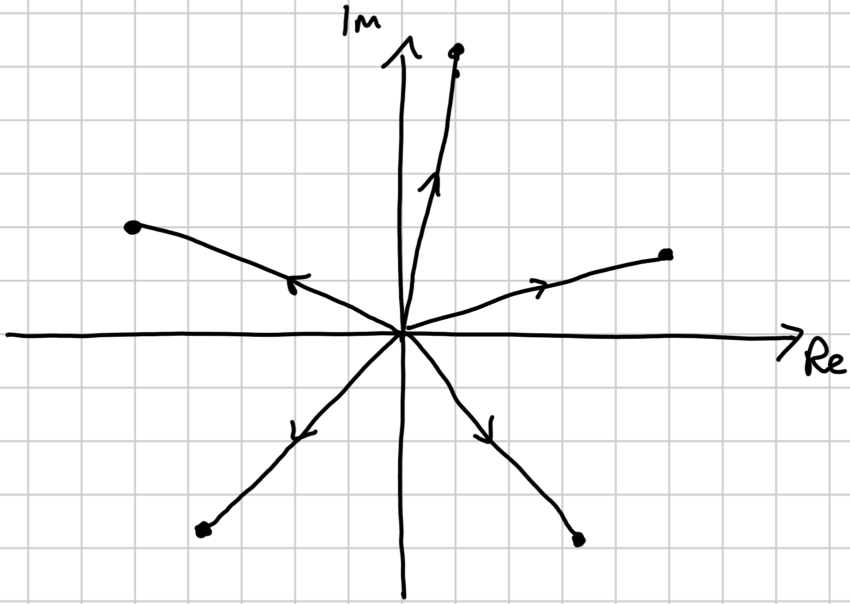
$\Rightarrow \quad r^5 = 32 \quad$ so $\quad r = 2$

and $\quad 5\theta = \frac{\pi}{3}$ or $\frac{\pi}{3} + 2\pi$ or $\frac{\pi}{3} + 4\pi$ or $\frac{\pi}{3} - 2\pi$ or $\frac{\pi}{3} - 4\pi$

$\theta = \frac{\pi}{15}$ or $\frac{7\pi}{15}$ or $\frac{13\pi}{15}$ or $-\frac{\pi}{3}$ or $-\frac{11\pi}{15}$

$z = 2e^{i\pi/15}$ or $2e^{i7\pi/15}$ or $2e^{i13/15}$ or $2e^{-i\pi/3}$ or $2e^{-i\frac{11\pi}{15}}$

On an Argand diagram :-

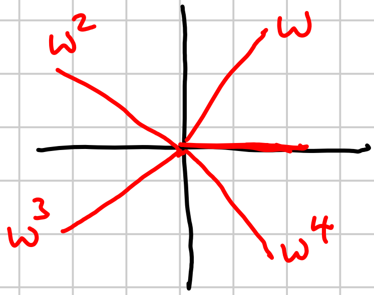


Note

① The n^{th} roots of any number, viewed as points on an Argand diagram, lie at the vertices of a regular n-gon. Viewed as vectors they form spokes at equally placed intervals

② The sum of the n^{th} roots of any number is 0.

③ The n^{th} roots of 1 have some additional properties :—



One root is 1

If n is even, another root is -1

The rest of the roots occur in complex conjugate pairs.

The roots can be written as.

$$1, \omega, \omega^2, \omega^3 \dots \omega^{n-1} \quad \text{where} \quad \omega = e^{i\frac{2\pi}{n}}$$

p37 Ex 3.2 Q 1c, 2b, 3, 4, 5, 7ab, 8b, 10