Further Integration

Standard Integrals and Methods

We need to know all the C4 methods and when to use them.
We know some additional standard derivatives :—

$f(x)$	$f'(x)$
$\sinh x$	$\cosh x$
$\cosh x$	$+\sinh x$
$\tanh x$	$\operatorname{sech}^2 x$
$\coth x$	$-\operatorname{cosech}^2 x$
$\operatorname{sech} x$	$-\operatorname{sech} x \tanh x$
$\operatorname{cosech} x$	$-\operatorname{cosech} x \coth x$

For the integrals of $\tanh x$ etc, just remember the method :—

$$\int \tanh x \, dx = \int \frac{\sinh x}{\cosh x} \, dx$$

Let $t = \cosh x \Rightarrow \dfrac{dt}{dx} = \sinh x \Rightarrow dx = \dfrac{dt}{\sinh x}$

$$I = \int \frac{1}{t} \, dt$$

$$= \ln|t|$$

$$= \underline{\underline{\ln|\cosh x| + c}}$$

$$\int \coth x \, dx = \ln|\sinh x| + c \qquad \text{(same method)}$$

$$\int \operatorname{sech} x \, dx = \int \frac{1}{\cosh x} \, dx$$

$$= \int \frac{2}{e^{x} + e^{-x}} \, dx$$

$$= \int \frac{2e^{x}}{e^{2x} + 1} \, dx$$

Let $t = e^{x} \Rightarrow \frac{dt}{dx} = e^{x} \Rightarrow dx = \frac{dt}{e^{x}}$

$$I = \int \frac{2}{t^2 + 1} \, dt$$

Let $t = \tan\theta \Rightarrow \frac{dt}{d\theta} = \sec^2\theta \qquad dt = \sec^2\theta \, d\theta$

$$I = \int \frac{2}{\cancel{\tan^2\theta + 1}} \cancel{\sec^2\theta} \, d\theta$$

$$= 2\theta$$

$$= 2\arctan t$$

$$= \underline{\underline{2\arctan(e^{x}) + c}}$$

For hyperbolic integrals, either use the corresponding trig method, or use the defintions in terms of e^{x}.

p58 Ex 4A Q 1 beg, 2 bcef

p61 Ex 4B Q 1 acd, 2abc, 3, 4, 5 bc, 6, 7, 8, 10, 11

Trig and Hyperbolic Substitutions

Example 1
$$\int \frac{1}{(x^2+9)^{3/2}}\,dx$$

Let $x = 3\sinh t \Rightarrow \dfrac{dx}{dt} = 3\cosh t$

$\Rightarrow dx = 3\cosh t\,dt$

$$I = \int \frac{1}{(9\sinh^2 t+9)^{3/2}}\,3\cosh t\,dt$$

$$= \int \frac{1}{(9\cosh^2 t)^{3/2}}\,3\cosh t\,dt$$

$$= \int \frac{1}{27\cosh^3 t}\,3\cosh t\,dt$$

$$= \frac{1}{9}\int \operatorname{sech}^2 t\,dt$$

$$= \frac{1}{9}\tanh t + c$$

$$= \underline{\underline{\frac{1}{9}\tanh\left(\operatorname{arsinh}\frac{x}{3}\right) + c}}$$

There are various ways of tackling similar questions. Also in the formula booklet there are 4 standard integrals of this type.

Example 2
$$\int \frac{1}{\sqrt{x^2-25}}\,dx$$

$$= \underline{\underline{\operatorname{arcosh}\left(\frac{x}{5}\right) + c}}$$

Example 3 $\displaystyle\int \frac{1}{25+9x^2}\,dx$

$$= \int \frac{1}{5^2+(3x)^2}\,dx$$

which is $f(3x)$ where $f(x) = \displaystyle\int \frac{1}{5^2+x^2}\,dx$.

$$= \frac{1}{3} \times \frac{1}{5} \arctan\left(\frac{3x}{5}\right) + C$$

$$= \frac{1}{15} \arctan\left(\frac{3x}{5}\right) + C$$

(or see p64 Example 11)

Example 4 $\displaystyle\int \frac{x+1}{\sqrt{x^2+1}}\,dx$

Split into :–

$$= \int \frac{x}{\sqrt{x^2+1}}\,dx + \int \frac{1}{\sqrt{x^2+1}}\,dx$$

(A) (B)

For (A) Let $t = x^2+1 \Rightarrow \dfrac{dt}{dx} = 2x$

$dx = \dfrac{dt}{2x}$

$$A = \int \frac{\cancel{x}}{\sqrt{t}} \frac{dt}{2\cancel{x}}$$

$$= \frac{1}{2} \int t^{-1/2}\,dt$$

$$= t^{1/2}$$

$$= \sqrt{x^2+1} + c$$

$B = \operatorname{arsinh} x$

So $\quad I = \sqrt{x^2+1} + \operatorname{arsinh} x + c$

Ex 4C @ 1, 4, 5, 7, 8, 9, 10, 11, 12, 13, 15, 16

Example 5

$$\int \frac{1}{\sqrt{x^2 - 6x + 13}} \, dx$$

Complete the square:

$$= \int \frac{1}{\sqrt{(x-3)^2 + 4}} \, dx$$

$$= \operatorname{arsinh}\left(\frac{x-3}{2}\right) + c$$

Ex 4D 1abceg, 2b, 4, 9, 10

Reduction Method of Integration

Example 1 If $I_n = \int_0^{\pi/2} \sin^n \theta \, d\theta$, show that

$$I_n = \frac{n-1}{n} I_{n-2}$$

Hence find $\int_0^{\pi/2} \sin^9 \theta \, d\theta$

$$I_n = \int_0^{\pi/2} \sin\theta \sin^{n-1}\theta \, d\theta$$

Let $u = \sin^{n-1}\theta$ and $\frac{dv}{d\theta} = \sin\theta$

$\frac{du}{d\theta} = (n-1)\sin^{n-2}\theta\cos\theta$ $\quad v = -\cos\theta$

$$I_n = \left[-\sin^{n-1}\theta\cos\theta\right]_0^{\pi/2} + (n-1)\int_0^{\pi/2} \sin^{n-2}\theta\cos^2\theta \, d\theta$$

$$I_n = 0 + (n-1)\int_0^{\pi/2} \sin^{n-2}\theta\,(1-\sin^2\theta)\, d\theta$$

$$I_n = (n-1)\left[\int_0^{\pi/2} \sin^{n-2}\theta \, d\theta - \int_0^{\pi/2} \sin^n\theta \, d\theta\right]$$

$$I_n = (n-1)\left[I_{n-2} - I_n\right]$$

$$I_n = (n-1)I_{n-2} - (n-1)I_n$$

$$nI_n = (n-1)I_{n-2}$$

$$I_n = \frac{n-1}{n} I_{n-2}$$ QED.

So $I_9 = \frac{8}{9} I_7$

$= \frac{8}{9}\frac{6}{7} I_5$

$= \frac{8}{9}\frac{6}{7}\frac{4}{5}\frac{2}{3} I_1$

$$I_1 = \int_0^{\pi/2} \sin\theta \, d\theta = \left[-\cos\theta\right]_0^{\pi/2}$$

$= 0 - -1 = \underline{1}$

$$\underline{\underline{I_9 = \frac{128}{315}}}$$

Example 2 If $I_n = \int_0^1 x^n \sqrt{1-x^3}\, dx$, show that $(2n+5) I_n = (2n-4) I_{n-3}$.

Hence find $\int_0^1 x^8 \sqrt{1-x^3}\, dx$

$$I_n = \int_0^1 x^{n-2}\, x^2 \sqrt{1-x^3}\, dx$$

Let $u = x^{n-2}$ $\qquad \frac{dv}{dx} = x^2 \sqrt{1-x^3}\, dx.$

$\frac{du}{dx} = (n-2)x^{n-3}$ $\qquad$ Let $t = 1-x^3 \Rightarrow \frac{dt}{dx} = -3x^2$

$$v = \int \cancel{x^2} \sqrt{t}\, \frac{dt}{-3\cancel{x^2}}$$

$$= -\frac{1}{3} \times \frac{2}{3} t^{3/2}$$

$$= -\frac{2}{9} (1-x^3)^{3/2}$$

$$I_n = \left[-\frac{2}{9} x^{n-2} (1-x^3)^{3/2} \right]_0^1 + \frac{2}{9}(n-2) \int_0^1 x^{n-3} (1-x^3)^{3/2}\, dx$$

$$I_n = 0 + \frac{2}{9}(n-2) \int_0^1 x^{n-3} (1-x^3) \sqrt{1-x^3}\, dx$$

$$= \frac{2}{9}(n-2) \int_0^1 (x^{n-3} - x^n) \sqrt{1-x^3}\, dx$$

$$I_n = \frac{2}{9}(n-2) \left[I_{n-3} - I_n \right]$$

$$9 I_n = (2n-4) I_{n-3} - (2n-4) I_n$$

$$(2n+5) I_n = (2n-4) I_{n-3}$$

QED

$$I_n = \frac{2n-4}{2n+5} I_{n-3}$$

$$I_8 = \frac{12}{21} I_5$$

$$= \frac{12}{21}\frac{6}{15} I_2$$

$$I_2 = \int_0^1 x^2 \sqrt{1-x^3}\, dx$$

$$= \left[-\frac{2}{9}\left(1-x^3\right)^{3/2}\right]_0^1$$

$$= 0 - \left(-\frac{2}{9}\right) = \frac{2}{9}$$

$$I_8 = \frac{12}{21}\frac{6}{15}\frac{2}{9} = \underline{\underline{\frac{16}{315}}}$$

We don't <u>always</u> use parts

<u>Example 3</u> If $I_n = \int_0^x \tan^n x\, dx$, show that

$$I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$

$$I_n = \int_0^x \tan^2 x \tan^{n-2} x\, dx$$

$$= \int_0^x (\sec^2 x - 1) \tan^{n-2} x\, dx$$

$$= \int_0^x \sec^2 x \tan^{n-2} x\, dx - I_{n-2}$$

Let $t = \tan x \Rightarrow \frac{dt}{dx} = \sec^2 x$

$$\Rightarrow dx = \frac{dt}{\sec^2 x}$$

$$I_n = \int t^{n-2}\, dt - I_{n-2}$$

$$= \left[\frac{\tan^{n-1} x}{n-1}\right]_0^x - I_{n-2}$$

$$I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2} \qquad \text{QED.}$$

Ex 4F Q. 1, 2, 7, 10, 13, 15