HYPERBOLIC FUNCTIONS

We know the power series for $\sin x$, $\cos x$ and e^x:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

From this we can confirm that

$$\frac{d}{dx}(\sin x) = 1 - \frac{3x^2}{3!} + \frac{5x^4}{5!} \cdots$$

$$= \cos x$$

and $\frac{d}{dx}(\cos x) = - \sin x$

and $\frac{d}{dx}(e^x) = e^x$

If we replace the alternating signs in the power series for $\sin x$ and $\cos x$, we obtain two new functions called hyperbolic sine and cosine:

$$\boxed{\begin{aligned} \sinh x &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \\ \cosh x &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots \end{aligned}}$$

These functions have many similarities with trig functions, but also with exponential functions.

We can see that

$$\boxed{\begin{aligned} &\frac{d}{dx}(\sinh x) = \cosh x \\ \text{AND } &\frac{d}{dx}(\cosh x) = \sinh x \end{aligned}}$$

Also and

$$\boxed{\begin{aligned} \cosh x + \sinh x &= e^x \\ \cosh x - \sinh x &= 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} - \cdots \\ &= e^{-x} \end{aligned}}$$

By adding these last two equations we obtain

$$2\cosh x = e^{x} + e^{-x}$$

$$\boxed{\begin{aligned}\cosh x &= \tfrac{1}{2}(e^{x} + e^{-x}) \\ \sinh x &= \tfrac{1}{2}(e^{x} - e^{-x})\end{aligned}}$$

and by subtracting,

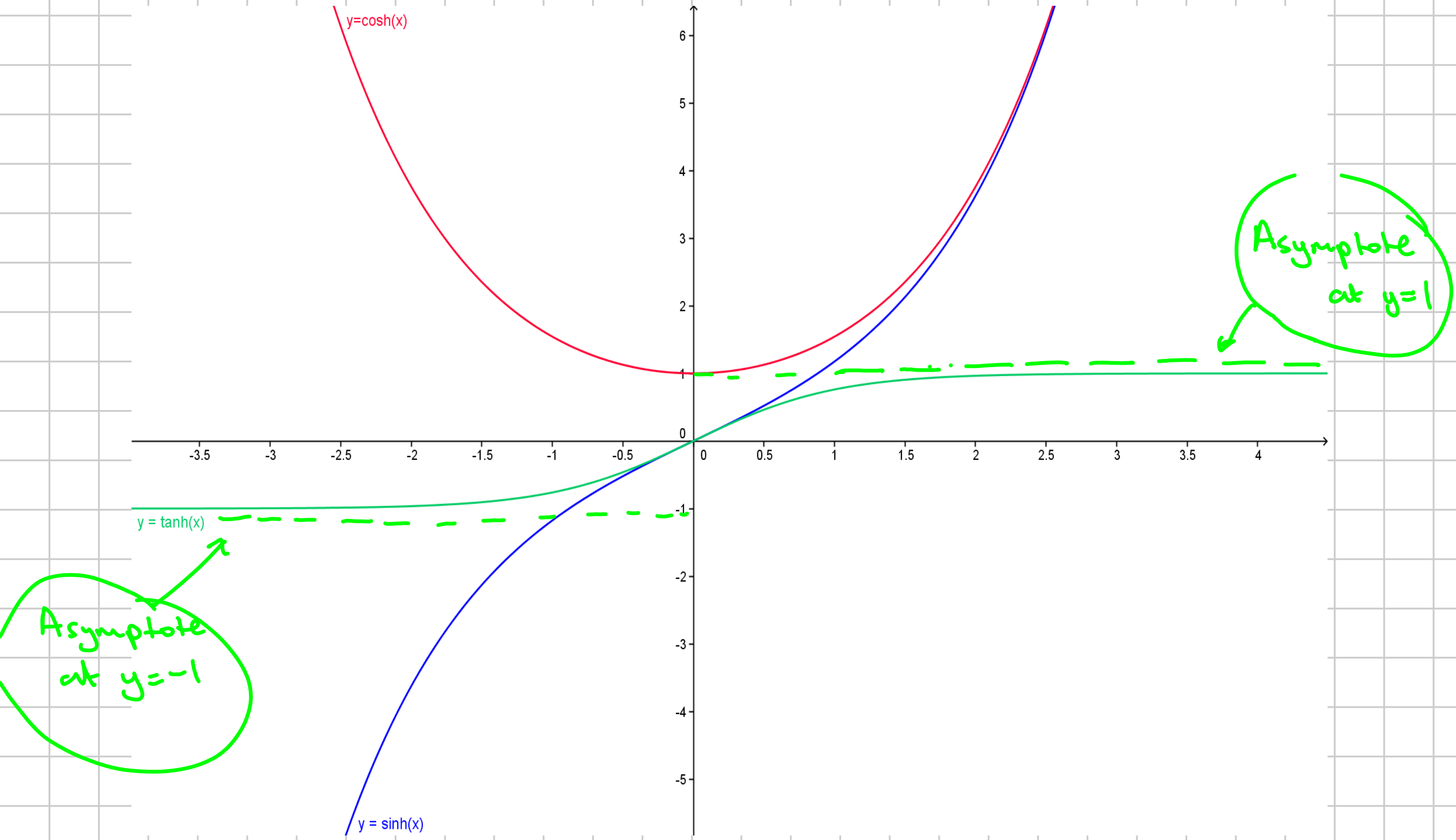
DEFINITIONS
- use in proofs

In the same way as for trig functions, we can define

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$$

and sech x, cosech x and coth x as well.

Graphs



Unlike the trig functions, hyperbolic functions are not PERIODIC.
But there are some similarities:

$\cosh 0 = \cos 0 = 1$ and $\cosh x$, like $\cos x$, is SYMMETRICAL about the y-axis.

$\sinh 0 = \sin 0 = 0$ and $\sinh x$, like $\sin x$, has ROTATIONAL SYMMETRY

Hyperbolic Identities

These look like trig identities, but to PROVE them use the definitions in terms of e^x

e.g ① Prove that $\sinh 2x = 2\sinh x \cosh x$

$$\text{RHS} = \cancel{2}\left(\frac{e^x - e^{-x}}{\cancel{2}}\right)\left(\frac{e^x + e^{-x}}{2}\right)$$

$$= \frac{1}{2}\left(e^{2x} + 1 - 1 - e^{-2x}\right)$$

$$= \text{LHS}$$

② Prove that $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

$$\text{RHS} = \left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y - e^{-y}}{2}\right)$$

$$= \frac{1}{4}\left[e^x e^y + \cancel{e^x e^{-y}} + \cancel{e^{-x} e^{y}} + e^x e^y - \cancel{e^x e^{-y}} - \cancel{e^{-x} e^y} + e^{-x} e^{-y}\right]$$

$$= \frac{1}{4}\left[2e^{x+y} + 2e^{-(x+y)}\right]$$

$$= \frac{1}{2}\left(e^{x+y} + e^{-(x+y)}\right)$$

$$= \text{LHS}$$

Note that ① is identical to the corresponding trig identity, but in ② the sign in the middle is changed. This illustrates OSBORN'S RULE, which says that

> Hyperbolic identities exactly match the trig equivalents, EXCEPT that if a term contains a product of two sines, the sign of that term is changed.

Notes:

① The product of sines may be $\sin^2 x$, or concealed as $\tan^2 x$ $\left(= \frac{\sin^2 x}{\cos^2 x}\right)$

② Of course Osborn's rule doesn't PROVE identities - just helps to remember them.

③ For the reason behind Osborn's rule, see notes on Complex Numbers II (later this term).

Equations

We can use identities to simplify, then convert to e^x form.

e.g. $$3\sinh 2x - 2\cosh^2 x = 2\cosh x$$

$$6\sinh x\cosh x - 2\cosh^2 x - 2\cosh x = 0$$
$$\cosh x\,(3\sinh x - \cosh x - 1) = 0$$

either $\cosh x = 0$ (not possible)

or $$\frac{3}{2}(e^x - e^{-x}) - \frac{1}{2}(e^x + e^{-x}) - 1 = 0$$

$$2e^x - 4e^{-x} - 2 = 0$$

(multiply by e^x) $$2e^{2x} - 4 - 2e^x = 0$$

(let $y = e^x$)

$$2y^2 - 2y - 4 = 0$$
$$y^2 - y - 2 = 0$$
$$(y-2)(y+1) = 0$$

$e^x = 2$ or $e^x = -1$ (not possible)

$$x = \ln 2$$

More derivatives

We have seen that $\frac{d}{dx}(\sinh x) = \cosh x$

$\frac{d}{dx}(\cosh x) = \sinh x$

Also, $$\frac{d}{dx}(\tanh x) = \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right)$$ (by quotient rule)

$$= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}$$ (but $\cosh^2 x - \sinh^2 x = 1$)
like $\cos^2 x + \sin^2 x = 1$

$$= \frac{1}{\cosh^2 x}$$

$$= \operatorname{sech}^2 x$$

Similarly, $\frac{d}{dx}(\coth x) = -\operatorname{cosech}^2 x.$

$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$

$\frac{d}{dx}(\operatorname{cosech} x) = -\operatorname{cosech} x \coth x$

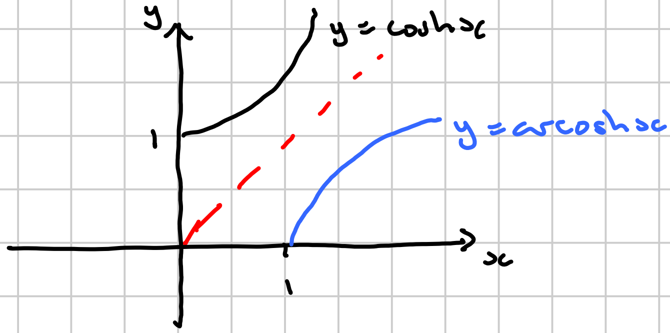
Same as trig except for one $-$ sign.
No need to learn these - if you need one, work it out.

Inverse hyperbolic functions

These are given the prefix "ar", rather than "arc" as for trig functions, e.g. $y = \sinh x \iff x = \operatorname{arsinh} y$
(or $x = \sinh^{-1} y$)

$\sinh x$ is a one-to-one function with domain $\mathbb{R}$ and range $\mathbb{R}$, so $\operatorname{arsinh} x$ is likewise

$\cosh x$ is many-to-one, so we need to restrict its domain to $x \geq 0$; the range is $\cosh x \geq 1$.
So $\operatorname{arcosh} x$ has domain $x \geq 1$, and range $\operatorname{arcosh} x \geq 0$



$\tanh x$ is one-to-one with domain $\mathbb{R}$ and range $-1 < \tanh x < 1$.
So $\operatorname{artanh} x$ has domain $-1 < x < 1$ and range $\mathbb{R}$

Just as hyperbolic functions can be written in terms of e^x, their inverse can be written in terms of $\ln x$.

e.g. $y = \operatorname{arcosh} x \Rightarrow x = \cosh y$

$$\Rightarrow x = \tfrac{1}{2}(e^y + e^{-y})$$

$$\Rightarrow e^y - 2x + e^{-y} = 0$$

(multiply by e^y) $\Rightarrow e^{2y} - 2x e^y + 1 = 0$

(complete the square) $\Rightarrow (e^y - x)^2 - x^2 + 1 = 0$

$$\Rightarrow e^y - x = \pm\sqrt{x^2-1}$$

$$\Rightarrow e^y = x \pm \sqrt{x^2-1}$$

But $x - \sqrt{x^2-1} < 1$, so this would make y negative. But we have defined $\operatorname{arcosh} x$ to be positive, so ....

$$\Rightarrow y = \ln(x + \sqrt{x^2-1})$$

$$\boxed{\operatorname{arcosh} x = \ln(x + \sqrt{x^2-1}) \quad (x \geq 1)}$$

Similarly we can show that $\boxed{\text{arsinh}\, x = \ln\left(x + \sqrt{x^2+1}\right) \quad (x \in \mathbb{R})}$

and $\boxed{\text{artanh}\, x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \quad (-1<x<1)}$

Example Solve $\cosh^2 x - 3\cosh x - 10 = 0$, giving the answer as a natural logarithm.

[We could convert $\cosh x$ into $\frac{1}{2}(e^x + e^{-x})$ as in the example above, but that would take longer.]

$$(\cosh x - 5)(\cosh x + 2) = 0$$

$\cosh x = 5$ or $\cosh x = -2$ (not possible)

$$x = \text{arcosh}\, 5$$
$$= \ln\left(5 + \sqrt{24}\right)$$

Derivatives of inverse trig and hyperbolic functions

To find these, use the rule $\boxed{\frac{dy}{dx} = \frac{1}{dx/dy}}$

e.g. $y = \arcsin x \Rightarrow x = \sin y$

$$\frac{dx}{dy} = \cos y$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

(but we want $\frac{dy}{dx}$ in terms of x,

so use $\cos^2 y + \sin^2 y = 1$

$\cos y = \sqrt{1 - \sin^2 y}$

$= \sqrt{1 - x^2}$)

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

Similarly, we can show that $\frac{d}{dx}(\arccos x) = \frac{-1}{\sqrt{1-x^2}}$

and (using $\sec^2 y = 1 + \tan^2 y$) $\frac{d}{dx}(\arctan x) = \frac{1}{1+x^2}$

For hyperbolic functions,

e.g $\quad y = \operatorname{arsinh} x \quad \Rightarrow \quad x = \sinh y$

$$\Rightarrow \frac{dx}{dy} = \cosh y$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cosh y}$$

But $\cosh^2 y - \sinh^2 y = 1$

$\Rightarrow \quad \cosh y = \sqrt{1 + \sinh^2 y}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$$

Similarly,

$$\frac{d}{dx}(\operatorname{arcosh} x) = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\operatorname{artanh} x) = \frac{1}{1-x^2}$$

[These are mainly useful in integration - see "Further Integration"]