Power Series

A power series is an "infinite polynomial"

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots\cdots$$

All well-behaved functions can be expressed as power series, valid for some interval around zero.

To find what the coefficients a_i are, we proceed as follows:—

$$f(0) = a_0$$

Differentiate

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \cdots$$

$$f'(0) = a_1$$

Carry on:

$$f''(x) = 2a_2 + 3\times 2\, a_3 x + 4\times 3\, a_4 x^2 + \cdots$$

$$f''(0) = 2a_2 \Rightarrow a_2 = \frac{f''(0)}{2!}$$

$$f'''(x) = 3\times 2\, a_3 + 4\times 3\times 2 a_4 x + \cdots$$

$$f'''(0) = 3!\, a_3 \Rightarrow a_3 = \frac{f'''(0)}{3!}$$

Continuing we find that

$$\boxed{f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!} + \cdots + \frac{f^{(n)}(0)}{n!}\,x^n + \cdots}$$

which is Maclaurin's Theorem.

Examples

① $\cos x = f(x).$

$$f(0) = 1$$

$$\begin{aligned} f'(x) &= -\sin x & f'(0) &= 0 \\ f''(x) &= -\cos x & f''(0) &= -1 \\ f'''(x) &= \sin x & f'''(0) &= 0 \\ f''''(x) &= \cos x & f''''(0) &= 1 \end{aligned}$$

etc

Hence $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} \dots\dots$

(valid for all x)

② $f(x) = e^x$

$f(0) = 1 = f'(0) = f''(0) = f'''(0)$ etc!

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

③ $f(x) = (1+x)^n$ $\qquad f(0) = 1$

$$\begin{aligned} f'(x) &= n(1+x)^{n-1} & f'(0) &= n \\ f''(x) &= n(n-1)(1+x)^{n-2} & f''(0) &= n(n-1) \\ f'''(x) &= n(n-1)(n-2)(1+x)^{n-3} & f'''(0) &= n(n-1)(n-2) \end{aligned}$$

$$\boxed{(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots}$$

which is the Binomial Theorem

In the formula book there are a number of given power series. We can often use these rather than starting from first principles.

Examples

(1) Use the given series to find the series for $\ln(\cos x)$ up to the term in x^4.

$$\ln(\cos x) = \ln\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots\dots\right)$$

$$= \ln\left(1 + \left(\frac{x^4}{4!} - \frac{x^2}{2!}\right)\right)$$

$$= \left(\frac{x^4}{4!} - \frac{x^2}{2!}\right) - \frac{\left(\frac{x^4}{4!} - \frac{x^2}{2!}\right)^2}{2} + \frac{\left(\frac{x^4}{4!} - \frac{x^2}{2!}\right)^3}{3} - \dots$$

$$= \left(\frac{x^4}{4!} - \frac{x^2}{2!}\right) - \frac{\left(\frac{x^8}{(4!)^2} - \frac{2x^6}{4!2!} + \frac{x^4}{(2!)^2}\right)}{2}$$

$$= \frac{x^4}{24} - \frac{x^2}{2} - \frac{x^4}{8}$$

$$= \underline{\underline{-\frac{x^2}{2} - \frac{x^4}{12} \dots\dots}}$$

(2) Write $\ln\left(\frac{1+x}{1-x}\right)$ as a power series as far as the term in x^5.

Write $\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} \dots\dots$$

$$\ln(1-x) = (-x) - \frac{(-x)^2}{2} + \frac{(-x)^3}{3} - \frac{(-x)^4}{4} + \frac{(-x)^5}{5} \dots$$

$$= -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} \dots$$

$$\ln\left(\frac{1+x}{1-x}\right) = 2x + \frac{2x^3}{3} + \frac{2x^5}{5} + \cdots\cdots$$

③ Use the power series for $\ln(1+x)$ to find $\ln(1.01)$ to 7 dp

$$\ln(1+0.01) = 0.01 - \frac{(0.01)^2}{2} + \frac{(0.01)^3}{3} - \cdots$$

$$= 0.01 - 0.00005 + 0.00000033$$

$$= 0.0099503$$

If x is small we often use approximations such as

$$\boxed{\begin{aligned} \sin x &\approx x \quad \left(\text{or} \quad x - \frac{x^3}{6}\right) \\ \cos x &\approx 1 - \frac{1}{2}x^2 \\ \tan x &\approx x \end{aligned}}$$

④ Find $\displaystyle\lim_{x\to 0}\left(\frac{\ln(1+x)}{3\sin x - x\cos x}\right)$

$$= \lim_{x\to 0}\left(\frac{x - \frac{x^2}{2} + \frac{x^3}{3}}{3\left(x - \frac{x^3}{3!}\right) - x\left(1 - \frac{x^2}{2!}\right)}\right)$$

$$= \lim_{x\to 0}\left(\frac{x - \frac{x^2}{2} + \frac{x^3}{3}}{3x - \frac{x^3}{2} - x + \frac{x^3}{2}}\right)$$

$$= \lim_{x\to 0}\left(\frac{x - \frac{x^2}{2} + \frac{x^3}{3}}{2x}\right)$$

$$= \lim_{x\to 0}\left(\frac{1}{2} - \frac{x}{4} + \frac{x^2}{6}\right)$$

$$= \underline{\underline{\frac{1}{2}}}$$

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Taylor Series

It is not always useful to have a series which valid for an interval centred on 0.

In this case we can use a series

$$f(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \cdots\cdots$$

By repeatedly substituting $x = c$, and differentiating, we find that

$$\boxed{f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 \cdots\cdots}$$

This is Taylor's series. If we put $c=0$, we get back to Maclaurin's series.

Example Find a series for $\sin x$ in ascending powers of $(x-\pi)$. Hence find an approximation to $\sin 3$ using 3 terms of your series.

$$\begin{aligned} f(x) &= \sin x & f(\pi) &= 0 \\ f'(x) &= \cos x & f'(\pi) &= -1 \\ f''(x) &= -\sin x & f''(\pi) &= 0 \\ f'''(x) &= -\cos x & f'''(\pi) &= 1 \end{aligned}$$

$$\sin x = -(x-\pi) + \frac{1}{3!}(x-\pi)^3 - \frac{1}{5!}(x-\pi)^5 + \cdots\cdots$$

$$\text{So } \sin 3 \approx -(3-\pi) + \frac{1}{6}(3-\pi)^3 - \frac{1}{120}(3-\pi)^5$$

$$\approx 0.14112000083$$

(calculator value for $\sin 3$ is $0.141120008\,1$!)

An alternative form of Taylor's series is

$$\boxed{f(c+x) = f(c) + f'(c)\,x + \frac{f''(c)}{2!}x^2 + \frac{f'''(c)}{3!}x^3 + \cdots}$$

For the above example this gives

$$\sin(\pi + x) = -x + \frac{1}{3!}x^3 - \frac{1}{5!}x^5 + \cdots\cdots$$

and to find $\sin 3$ we need $x = -0.14159\ldots\ldots$

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