Note Title 12/07/2010

Numerical Methods for Solving Equations

Most equations cannot be solved exactly, so we need methods for finding approximate solutions.

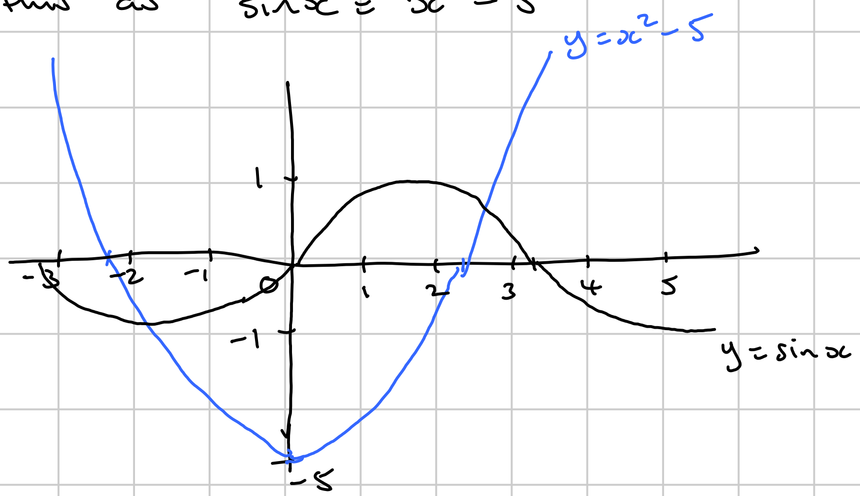
Steps involved :—

- Sketch graphs to see how many solutions there are.
- Look for a sign change to find an interval containing a root
- Various iterative methods for improving the accuracy of a solution :—
  - "$x = g(x)$" method (covered in C3)
  - interval bisection
  - linear interpolation
  - the Newton-Raphson method

  } covered in the chapter.
- Check that the solution has the required degree of accuracy (using a sign change again on the upper and lower bound).

Example Consider the equation $\sin x - x^2 + 5 = 0$

Write this as $\sin x = x^2 - 5$



Equation has 2 solutions

Let $f(x) = \sin x - x^2 + 5$.

$\boxed{\text{NB use RADIANS}}$

$f(2) = 1.9\ldots$ +ve

$f(3) = -3.9\ldots$ −ve

} sign change $\Rightarrow$ root between 2 and 3

Now we need to improve the accuracy of our solution using an iterative method.

Interval Bisection

With this method we keep halving the interval within which the sign change occurs

Initial interval $(a_0, b_0) = (2, 3)$

Midpoint $x_1 = \frac{a_0 + b_0}{2} = 2.5$

$f(2.5) = -0.65$ (−ve so 2.5 replaces 3)

So new interval is $(a_1, b_1) = (2, 2.5)$

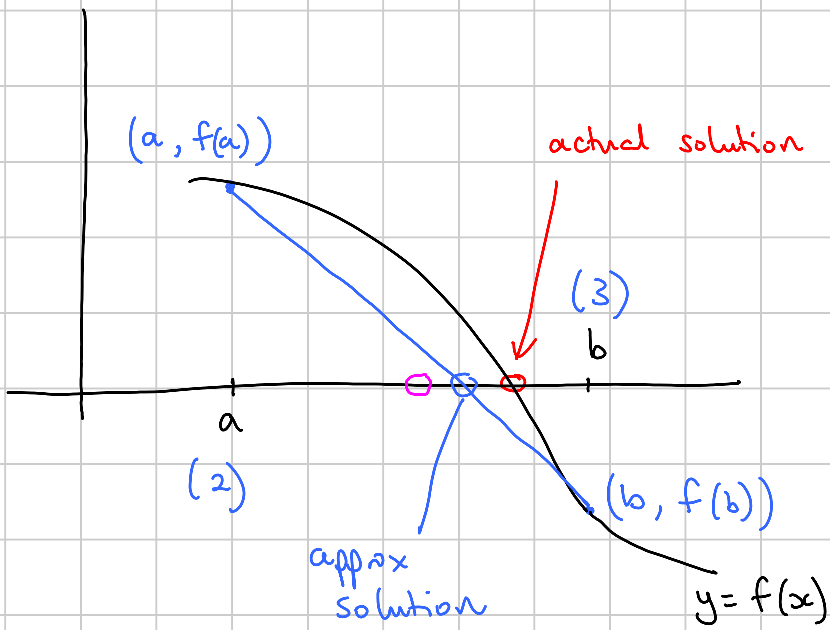
Midpoint $x_2 = \frac{a_1 + b_1}{2} = 2.25$

$f(2.25) = 0.72$ (+ve so 2.25 replaces 2)

So new interval is $(2.25, 2.5)$

etc

Linear Interpolation



The bisection method only takes account of the <u>sign</u> of $f(a)$ and $f(b)$, not their relative magnitudes.

Linear interpolation improves on this by using a straight line as an approximation to the curve between $(a, f(a))$ and $(b, f(b))$.

Gradient of line $= \dfrac{f(b) - f(a)}{b - a} = \dfrac{-3.858\ldots - 1.909\ldots}{3 - 2}$

$= -5.767$

Equation of line $\quad y - 1.909.. = -5.767(x - 2)$

Root is where $y = 0 \Rightarrow x = \dfrac{-1.909}{-5.767} + 2$

$= 2.331\ldots$

So $x_1 = 2.331\ldots$

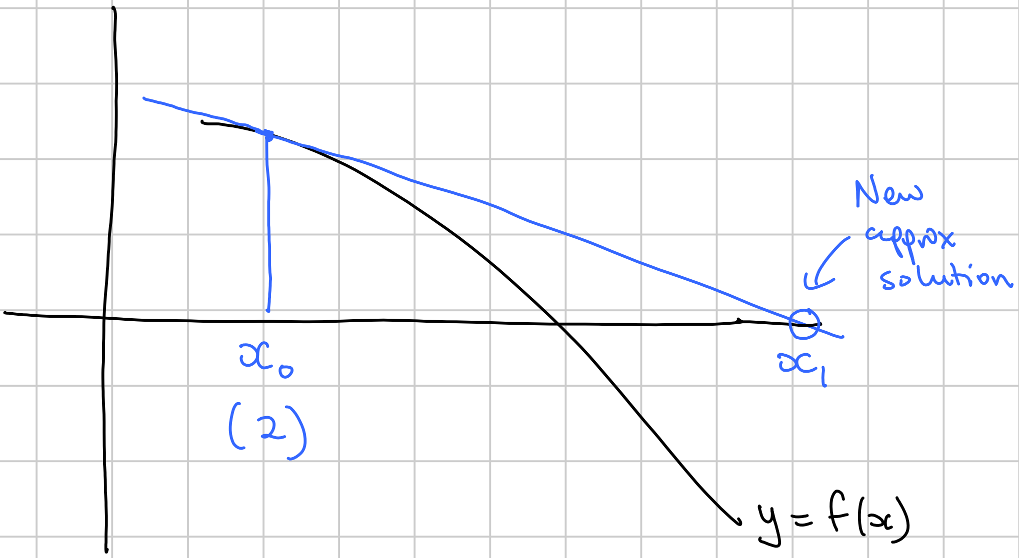
$f(x_1) = 0.291 > 0 \quad$ so 2.331 replaces 2

New interval (a_1, b_1) is $(2.331, 3)$

We can now repeat the process on this interval.

<u>Newton-Raphson Method</u>

This method only requires one initial value close to the root rather than an interval containing the root.



It approximates the curve using the <u>tangent</u> at the initial approx (x_0).

To find x_1:

Gradient of tangent is $f'(x_0)$

Equation of tangent is $y - f(x_0) = f'(x_0)(x - x_0)$

Solution is at $(x_1, 0)$ $\quad 0 - f(x_0) = f'(x_0)(x_1 - x_0)$

$$\frac{-f(x_0)}{f'(x_0)} = x_1 - x_0$$

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

(which is in the formula book)

In our example $\quad f(x) = \sin x - x^2 + 5$

$\quad f'(x) = \cos x - 2x$

So if $x_0 = 2$, $\quad x_1 = 2 - \frac{f(2)}{f'(2)} = 2.432\ldots$

p63	Ex 3.2	Q 5, 9
p66	Ex 3.3	Q 5, 6
p70	Ex 3.4	Q 3, 7, 8