

# COMPLEX NUMBERS

Note Title

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These are a natural extension of the number system so that we can find the square root of any number. They form a COMPLETE system - no further extension is necessary.

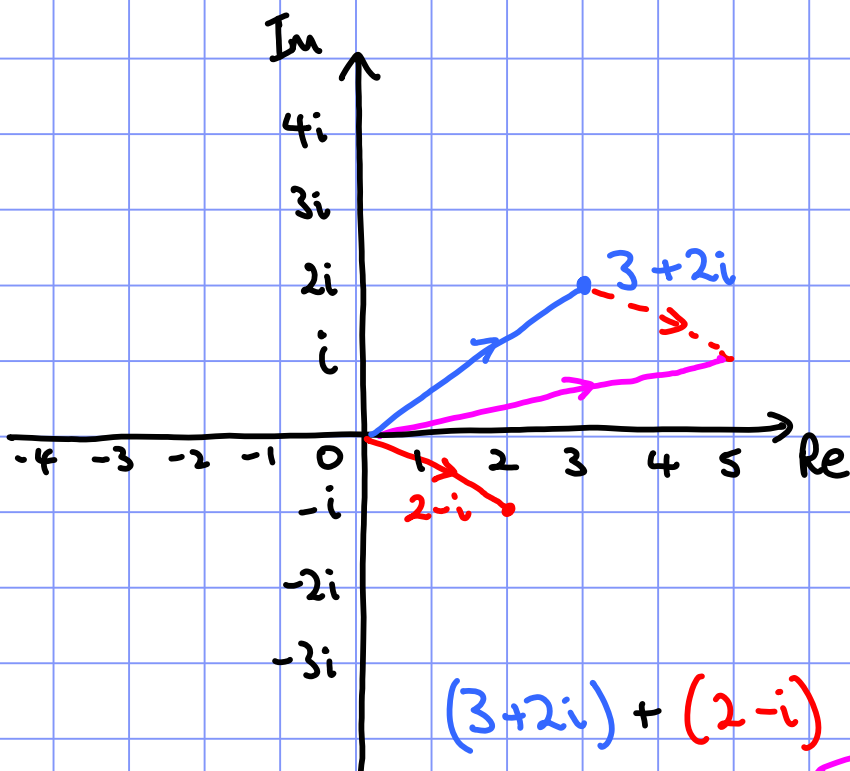
$$\mathbb{N} \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{R} \rightarrow \mathbb{C}$$

We write  $\sqrt{-1} = i$ .

$$\text{So } \sqrt{-4} = \sqrt{4 \times -1} = \sqrt{4} \times \sqrt{-1} = 2i$$

$$\sqrt{-2} = \sqrt{2 \times -1} = \sqrt{2} \times \sqrt{-1} = i\sqrt{2} \quad \text{etc.}$$

Real numbers completely fill the number line; complex numbers are represented using a number plane called the 'Argand Diagram'



A number is represented either by a point, or by the vector from 0 to that point.

Adding and subtracting complex numbers corresponds to adding/subtracting their vectors

$$(3+2i) + (2-i) = 5+i$$

## Multiplying and dividing

$$\textcircled{1} \quad (2+3i)(4-8i) = 8 + 12i - 16i - 24i^2 \\ = \underline{\underline{32 - 4i}}$$

$$\textcircled{2} \quad i^3 = i^2 \times i \\ = -1 \times i \\ = \underline{\underline{-i}}$$

$$i^4 = \underline{\underline{1}} \quad i^5 = i^4 \times i \quad \underline{\underline{\text{etc}}} \\ = \underline{\underline{i}}$$

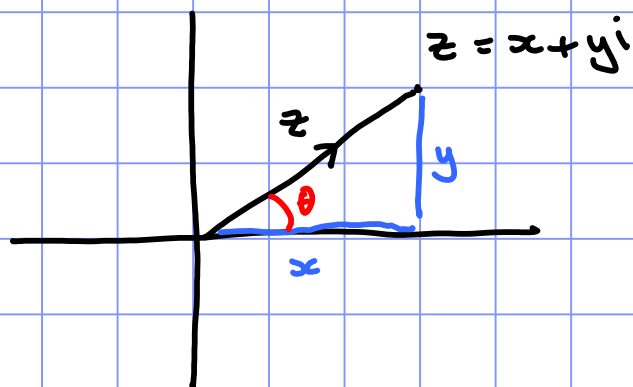
The **COMPLEX CONJUGATE** of a complex number  $z (= a+bi)$  is  $z^* = a-bi$ .  
Note that both  $z + z^*$  and  $zz^*$  are always real numbers.

To divide two complex numbers we need to 'realize the denominator' by multiplying top and bottom by the complex conjugate.

$$\textcircled{3} \quad \frac{2+3i}{5-2i} \\ = \frac{2+3i}{5-2i} \times \frac{5+2i}{5+2i} \\ = \frac{10 + 15i + 4i + 6i^2}{25 - \cancel{10i} + \cancel{10i} - 4i^2} = \frac{4 + 19i}{29} \\ = \underline{\underline{\frac{4}{29} + \frac{19}{29}i}}$$

# Modulus and Argument of a Complex Number

The modulus of a complex number is the length of its vector. It is written  $|z|$

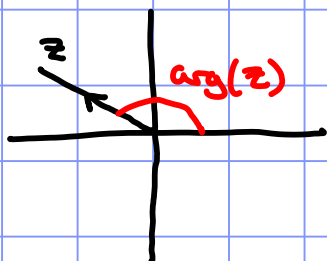


So  $|z| = \sqrt{x^2 + y^2}$

The argument of a complex number is the angle its vector makes with the positive  $x$ -axis ( $\theta$  on the diagram above). It is written  $\arg(z)$

So  $\arg(z) = \arctan\left(\frac{y}{x}\right)$  but take account of quadrant

Example If  $z = -\sqrt{3} + i$ , find  $\arg(z)$



$$\arctan\left(\frac{1}{-\sqrt{3}}\right) = -\frac{\pi}{6}$$

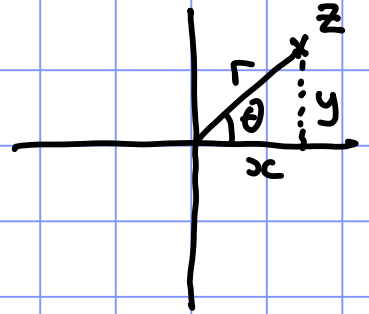
$$\text{but } \arg(z) = -\frac{\pi}{6} + \pi = \frac{5\pi}{6}$$

## Writing numbers in modulus - argument form

The usual way of writing a complex number is  $z = x + yi$

However, since

$$\begin{aligned}x &= r \cos \theta \\y &= r \sin \theta\end{aligned}$$



we can write  $z = r \cos \theta + i r \sin \theta$

$$z = r (\cos \theta + i \sin \theta)$$

$r = \text{modulus}$   
 $\theta = \text{argument}$

which is the modulus - argument form of  $z$ .

This is useful for multiplying complex numbers:-

$$\text{if } z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$$

$$z_2 = r_2 (\cos \theta_2 + i \sin \theta_2)$$

$$\begin{aligned}\text{then } z_1 z_2 &= r_1 r_2 (\cos \theta_1 + i \sin \theta_1) (\cos \theta_2 + i \sin \theta_2) \\&= r_1 r_2 \left[ (\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) \right. \\&\quad \left. + i (\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2) \right]\end{aligned}$$

$$= r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$$

$$\text{So } |z_1 z_2| = r_1 r_2 \quad \text{and} \quad \arg(z_1 z_2) = \theta_1 + \theta_2$$

$$|z_1 z_2| = |z_1| |z_2| \quad \arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

$$\begin{aligned}
 \text{Also } \frac{z_1}{z_2} &= \frac{r_1 (\cos \theta_1 + i \sin \theta_1)}{r_2 (\cos \theta_2 + i \sin \theta_2)} \times \frac{(\cos \theta_2 - i \sin \theta_2)}{(\cos \theta_2 - i \sin \theta_2)} \\
 &= \frac{r_1 \left[ (\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2) + i (\sin \theta_1 \cos \theta_2 - \cos \theta_1 \sin \theta_2) \right]}{r_2 (\cos^2 \theta_2 + \sin^2 \theta_2)} \\
 &= \frac{r_1}{r_2} \left[ \cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2) \right]
 \end{aligned}$$

$$\text{So } \left| \frac{z_1}{z_2} \right| = \frac{r_1}{r_2} \quad \text{and} \quad \arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2.$$

Summarizing,

$$|z_1 z_2| = |z_1| |z_2| \quad \arg(z_1 z_2) = \arg z_1 + \arg z_2$$

$$\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|} \quad \arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$$

(rules for arg are same as for log)

Example

Simplify

$$\frac{6 (\cos \frac{\pi}{3} + i \sin \frac{\pi}{3})}{2 (\cos \frac{\pi}{6} - i \sin \frac{\pi}{6})}$$

$$= \frac{6 (\cos \frac{\pi}{3} + i \sin \frac{\pi}{3})}{2 (\cos(-\frac{\pi}{6}) + i \sin(-\frac{\pi}{6}))}$$

$$= 3 (\cos(\frac{\pi}{3} - (-\frac{\pi}{6})) + i \sin \frac{\pi}{2})$$

$$= \underline{\underline{3i}}$$

## Equating Real and Imaginary Parts

If  $z = x + yi$  and  $w = u + vi$  and we know that  $z = w$ , then we often find it useful to split this into two equalities involving real numbers.

$$\left. \begin{array}{l} \text{Re}(z) = \text{Re}(w) \\ \text{Im}(z) = \text{Im}(w) \end{array} \right\} \begin{array}{l} \text{i.e. } x = u \\ \text{i.e. } y = v \end{array}$$

One application of this is in finding square roots.

Example Find  $\sqrt{5+12i}$

Suppose  $z^2 = 5 + 12i$

Write  $z$  as  $x + yi$ , so

$$(x+yi)(x+yi) \quad (x+yi)^2 = 5 + 12i$$

$$\underline{(x^2 - y^2)} + \underline{2x yi} = \underline{5} + \underline{12i}$$

Equating real and imaginary parts

$$\left. \begin{array}{l} \textcircled{1} \quad x^2 - y^2 = 5 \\ \textcircled{2} \quad 2xy = 12 \end{array} \right\}$$

$$\textcircled{2} \Rightarrow y = \frac{6}{x}$$

Subst in  $\textcircled{1} \Rightarrow x^2 - \frac{36}{x^2} = 5$

$$x^4 - 5x^2 - 36 = 0$$
$$(x^2 + 4)(x^2 - 9) = 0$$

$x^2 + 4 = 0$  has no solutions, since  $x \in \mathbb{R}$ .

$$x^2 - 9 = 0 \Rightarrow \left. \begin{array}{l} x = 3 \\ y = 2 \end{array} \right\} \text{ or } \left. \begin{array}{l} x = -3 \\ y = -2 \end{array} \right\}$$

So  $z = 3 + 2i$  or  $-3 - 2i$

i.e.,  $z = \pm (3 + 2i)$

## Complex Roots of Polynomial Equations

The "Fundamental Theorem of Algebra" states that if  $P(x)$  is a polynomial of degree  $n$ , i.e.,

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$$

( $a_i$  are constants,  $a_i \in \mathbb{C}$ ,  $a_n \neq 0$ )

then the equation  $P(x) = 0$  has  $n$  solutions

$$\alpha_1, \alpha_2, \dots, \alpha_n$$

( $\alpha_i \in \mathbb{C}$ , some may be repeated roots)

It follows that  $P(x)$  can be factorized into  $n$  linear factors

$$P(x) = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \dots (x - \alpha_n)$$

Example Solve the equation

$$x^2 + 2x + 5 = 0$$

and hence factorize  $x^2 + 2x + 5$ .

$$x = \frac{-2 \pm \sqrt{4 - 20}}{2}$$

$$= \frac{-2 \pm 4i}{2}$$

$$x = \underline{-1 + 2i} \quad \text{or} \quad \underline{-1 - 2i}$$

$$\begin{aligned} \text{So } x^2 + 2x + 5 &= (x - (-1 + 2i))(x - (-1 - 2i)) \\ &= (x + 1 - 2i)(x + 1 + 2i) \end{aligned}$$

It can be shown that if all the coefficients of a polynomial  $P(x)$  are real ( $a_i \in \mathbb{R}$  above) then any complex roots of  $P(x) = 0$  occur in complex conjugate pairs. (ie if  $\alpha$  is one root,  $\alpha^*$  is another root)  
(It follows that any equation of ODD order must have at least one REAL root, and an ODD number of them.)

Example Solve the equation

$$x^4 - 6x^3 + 6x^2 + 34x - 195 = 0$$

given that one of the roots is  $x = 2 - 3i$ .

Another root is  $2 + 3i$

So two of the factors are



$$(x - (2 + 3i)) \text{ and } (x - (2 - 3i))$$

So a quadratic factor is

$$\left( x^2 - (2 + 3i)x - (2 - 3i)x + 13 \right)$$

ie,  $x^2 - 4x + 13$

$$\begin{array}{r} x^2 - 2x - 15 \\ x^2 - 4x + 13 \overline{) x^4 - 6x^3 + 6x^2 + 34x - 195} \\ \underline{x^4 - 4x^3 + 13x^2} \phantom{- 195} \\ -2x^3 - 7x^2 + 34x \phantom{- 195} \\ \underline{-2x^3 + 8x^2 - 26x} \phantom{- 195} \\ -15x^2 + 60x - 195 \\ \underline{-15x^2 + 60x - 195} \\ 0 \end{array}$$

So the other quadratic factor is  $x^2 - 2x - 15$   
 $= (x - 5)(x + 3)$

So all 4 solutions are

$$\underline{\underline{x = 2 + 3i, 2 - 3i, 5 \text{ or } -3}}$$

and the factors are

$$(x - 2 - 3i)(x - 2 + 3i)(x - 5)(x + 3) = 0$$

(four linear factors)

or  $(x^2 - 4x + 13)(x - 5)(x + 3) = 0$

(which is factorized as far as possible using only real numbers)

p 39 Ex 2.6 Q 3 a c e g h, 8, 9

p 44 Ex 2.7 Q 3, 4, 5, 6, 7, 8, 12

p 48 Ex 2.8 Q 4,

p 51 Ex 2.9 Q 2, 5, 7