Proof by Induction

This is a method for proving that a certain result is true for all positive integers n (ie, for $n \in \mathbb{N}$). There are two steps:

1. We show that the result is true for $n = 1$
2. We ASSUME the result is true for $n = k$, and show that it follows that it is true for $n = k+1$

Examples

① Prove that $7^n + 2^{2n} + 1$ is divisible by 6 for all $n \in \mathbb{N}$.

Try $n = 1$: $7^1 + 2^2 + 1 = 12 = 2 \times 6$

Assume this is true for $n = k$ $(k \geqslant 1)$
ie, that $7^k + 2^{2k} + 1 = 6a$ $(a \in \mathbb{N})$

Now consider $n = k+1$

$$
\begin{aligned}
7^{k+1} + 2^{2k+2} + 1 &= 7(7^k) + 4(2^{2k}) + 1 \\
&= 7^k + 2^{2k} + 1 + 6(7^k) + 3(2^{2k}) \\
&= 6a + 6(7^k) + 6(2^{2k-1}) \\
&= 6[a + 7^k + 2^{2k-1}] \\
&= 6b \quad (b \in \mathbb{N})
\end{aligned}
$$

Hence by induction the result is true for all $n \in \mathbb{N}$.

② Prove that $1^3 + 2^3 + \ldots n^3 = \dfrac{n^2(n+1)^2}{4}$ for $n \in \mathbb{N}$.

Try $n = 1$: $1^3 = 1 = \dfrac{1^2 \times 2^2}{4}$ which is true.

Now assume result is true for $n = k$
ie, $1^3 + 2^3 + \ldots + k^3 = \dfrac{k^2(k+1)^2}{4}$

Consider $n = k+1$

$$1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} + (k+1)^3$$
$$= \frac{1}{4}(k+1)^2\left[k^2 + 4(k+1)\right]$$
$$= \frac{1}{4}(k+1)^2(k^2 + 4k + 4)$$
$$= \frac{(k+1)^2(k+2)^2}{4}$$

which is the formula with $n = k+1$

Hence by induction the result is true for all $n \in \mathbb{N}$.

(3) A sequence is defined by:

$u_1 = 1$, $u_2 = 5$, $u_{n+2} = 5u_{n+1} - 6u_n$ $(n \geq 1)$

[ie, 1, 5, 19, 65, ...]

Prove by induction that $u_n = 3^n - 2^n$

Try $n = 1$ $3^1 - 2^1 = 1 = u_1$
and $n = 2$ $3^2 - 2^2 = 5 = u_2$

Assume the result is true for $n = k$ and $n = k+1$

ie, that $u_k = 3^k - 2^k$
and $u_{k+1} = 3^{k+1} - 2^{k+1}$

Then, from the given definition,

$$u_{k+2} = 5\left(3^{k+1} - 2^{k+1}\right) - 6\left(3^k - 2^k\right)$$
$$= 3^k(15 - 6) + 2^k(6 - 10)$$
$$= 9(3^k) - 4(2^k)$$
$$= 3^2 \times 3^k - 2^2 \times 2^k$$
$$= 3^{k+2} - 2^{k+2}$$

which is the result for $n = k+2$

Hence by induction the result is true for all $n \in \mathbb{N}$.

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