Differential Equations

Note Title 04/03/2013

These are equations involving derivatives. They arise in many situations where RATES OF CHANGE are involved.

Example Newton's law of cooling states that the rate at which the temperature (θ) of a body is falling is proportional to the difference between θ and the temperature of the surroundings (which is taken as a constant, c).
Write this as a differential equation (DE).

$$\frac{d\theta}{dt} \propto \theta - c$$

$$\Rightarrow \quad \frac{d\theta}{dt} = -k(\theta - c) \qquad (\text{where } k > 0)$$

To solve a DE we need to eliminate the derivative. Since this involves integration, an arbitrary constant is introduced. The solution in this form is called a GENERAL SOLUTION.

If we are given some INITIAL CONDITIONS we can substitute them in to find a value for the constant, and thus find the PARTICULAR SOLUTION for these conditions

There are many techniques for solving DEs but for C4 we only need to know one - SEPARATING THE VARIABLES
To do this we get all 'x's (including the 'dx') on one side and all 'y's on the other, and integrate both sides.

Examples

① If $x^2 \frac{dy}{dx} = \cos^2 y$

(a) Find the general solution
(b) Find the solution for which $y = 0$ when $x = 1$

$$x^2 \, dy = \cos^2 y \, dx$$

$$dy = \frac{\cos^2 y}{x^2} \, dx$$

$$\int \frac{1}{\cos^2 y}\,dy = \int \frac{1}{x^2}\,dx$$

$$\int \sec^2 y\,dy = \int x^{-2}\,dx$$

$$\tan y = -x^{-1} + c$$

$$y = \arctan\left(c - \frac{1}{x}\right)$$

(b) If $y=0$ when $x=1$,

$$0 = \arctan(c-1)$$

$$\tan 0 = c - 1$$

$$0 = c - 1$$

$$c = 1$$

$$y = \arctan\left(1 - \frac{1}{x}\right)$$

(2) In Newton laws of Cooling, suppose the temperature of the room is 16°, and that $k = \frac{1}{2}$, so that the DE is

$$\frac{d\theta}{dt} = -\frac{1}{2}(\theta - 16)$$

Given that $\theta = 24$ when $t=0$, find an expression for θ in terms of t and sketch it.

$$d\theta = -\frac{1}{2}(\theta - 16)\,dt$$

$$\int \frac{1}{\theta - 16}\,d\theta = \int -\frac{1}{2}\,dt$$

$$\ln(\theta - 16) = -\frac{1}{2}t + c$$

$$\theta - 16 = e^{-\frac{1}{2}t + c}$$

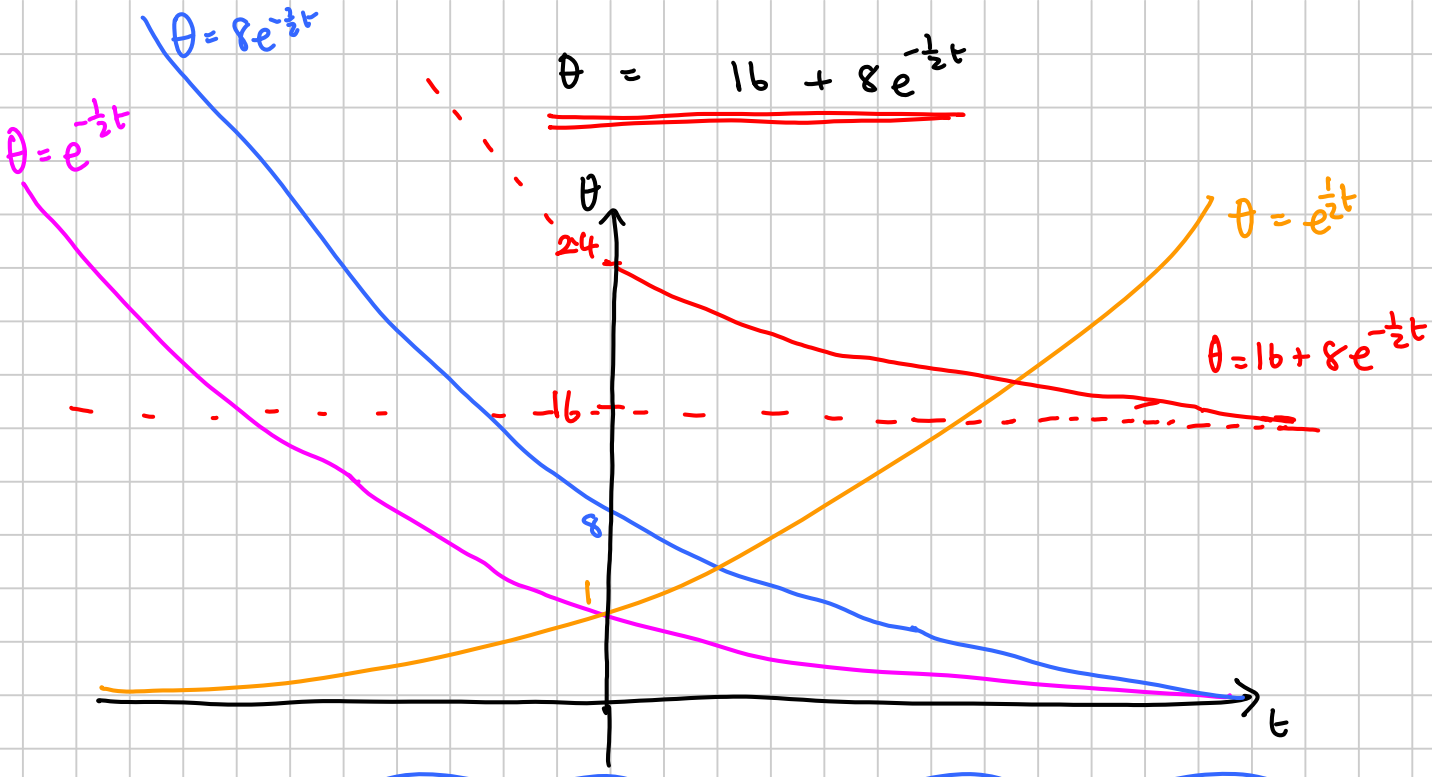
$$\theta - 16 = e^{-\frac{1}{2}t} \times e^{c}$$ ← still an arbitrary constant - call it A

$$\theta - 16 = Ae^{-\frac{1}{2}t}$$

$$\theta = 16 + Ae^{-\frac{1}{2}t}$$

Subst $\theta = 24$ when $t=0$ $\Rightarrow$ $24 = 16 + Ae^0$

$$A = 8$$



p41 Ex 4E Q 1, 3, 4, 5, 6, 7, [9, 10, 11] (← words to symbols)

p 110 Ex 6J Q 1 abd, 4 ade (just solve eqns)

p 111 Ex 6K Q 1, 2, 3, 5, 6, 7 (words AND solve eqns)