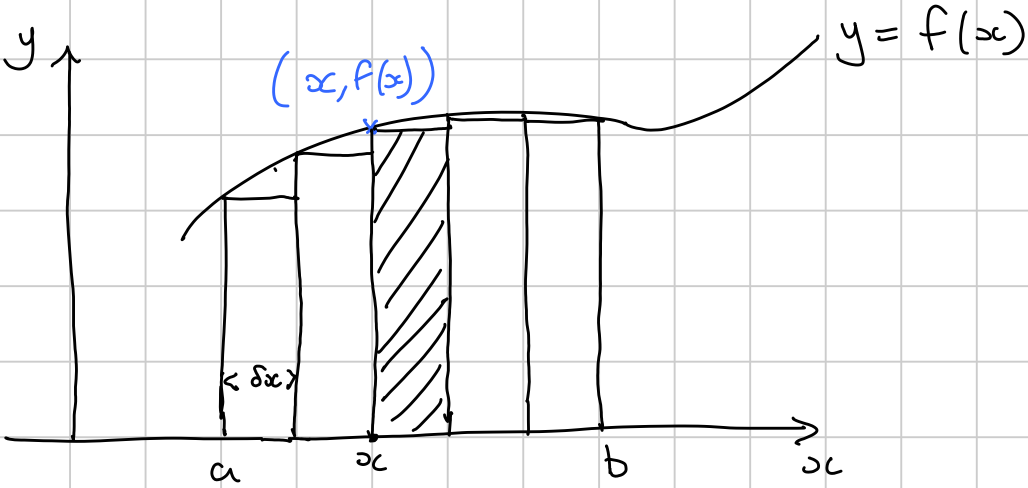
Using integration to find areas and volumes

An integral can be viewed as the limit of a sum of infinitely many infinitesimal quantities.

ie,
$$\boxed{\lim_{\delta x \to 0} \sum_{x=a}^{x=b} f(x)\,\delta x = \int_a^b f(x)\,dx}$$

e.g. if we wish to find an area between a curve the x-axis and the lines $x=a$ and $x=b$,



We can divide it into strips of width δx. Each strip can be approximated by a rectangle with area $f(x)\,\delta x$.

So the total area is approximately
$$\sum_{x=a}^{x=b} f(x)\,\delta x$$

If we use a finite number of strips, this answer is only approximate (and we would be better using trapezia rather than rectangles). However, if we let $\delta x \to 0$ so that we have an infinite number of infinitesimally wide strips, the area becomes exact.

So the area is exactly

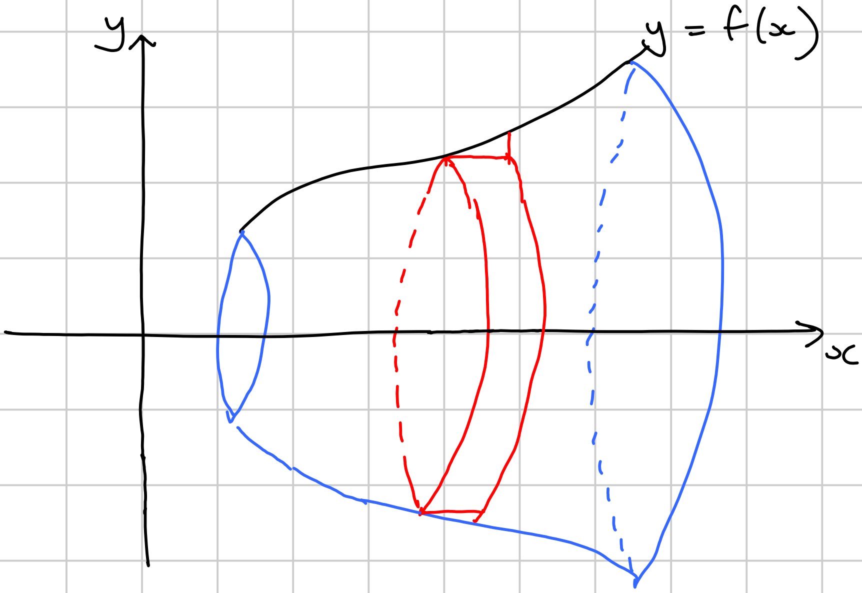
$$\lim_{\delta x \to 0} \sum_{x=a}^{x=b} f(x)\, \delta x$$

which is equal to

$$\int_{x=a}^{x=b} f(x)\, dx$$

Volume of a solid of revolution

If we rotate an area around the x-axis, we get a solid of revolution.



A thin strip of this solid is approximately a cylinder of radius $f(x)$ and 'height' δx

So its volume is $\pi \left[f(x)\right]^2 \delta x$

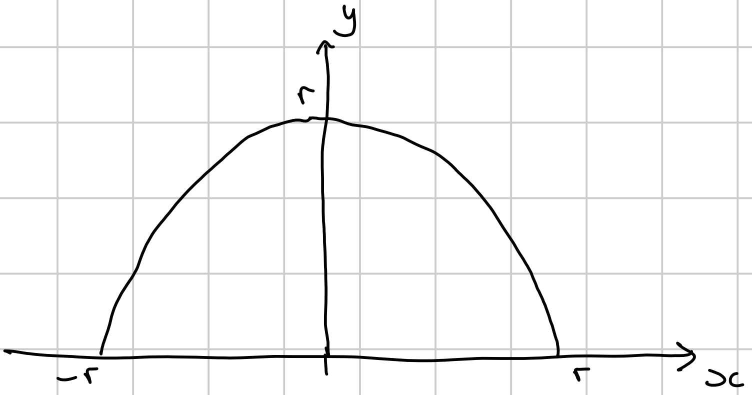
So the volume of the whole solid is

$$\lim_{\delta x \to 0} \left(\sum_{x=a}^{x=b} \pi \left[f(x) \right]^2 \delta x \right)$$

which is

$$\boxed{\pi \int_{x=a}^{x=b} \left[f(x) \right]^2 dx} \quad \text{or} \quad \boxed{\pi \int_{x=a}^{x=b} y^2 \, dx}$$

Examples ① Find the volume of the solid created when the area under the semicircle $y = \sqrt{r^2 - x^2}$ is rotated through 2π about the x-axis.



$$\text{Volume} = \int_{-r}^{r} \pi \left[f(x) \right]^2 dx$$

$$= \pi \int_{-r}^{r} r^2 - x^2 \, dx$$

$$= \pi \left[r^2 x - \frac{1}{3} x^3 \right]_{-r}^{r}$$

$$= \pi \left[\left(r^3 - \frac{1}{3} r^3 \right) - \left(-r^3 + \frac{1}{3} r^3 \right) \right]$$

$$= \frac{4}{3} \pi r^3$$

Note Title 21/04/2010

(2) Find the volume of the solid formed when the curve

$$\left.\begin{aligned} x &= t^2 \\ y &= t^3 - 4t \end{aligned}\right\}$$

between the limits $t = 0$ and $t = 2$ is rotated through 2π about the x-axis.

$$\text{Volume} = \int \pi y^2 \, dx$$

$$= \int \pi y^2 \frac{dx}{dt} \, dt$$

$$= \int_{t=0}^{t=2} \pi (t^3 - 4t)^2 \, 2t \, dt$$

$$= \pi \int_{t=0}^{t=2} (t^6 - 8t^4 + 16t^2) \, 2t \, dt$$

$$= \pi \int_{t=0}^{t=2} 2t^7 - 16t^5 + 32t^3 \, dt$$

$$= \pi \left[\frac{1}{4} t^8 - \frac{8}{3} t^6 + 8t^4 \right]_0^2$$

$$= \pi \left[64 - \frac{512}{3} + 128 \right]$$

$$= \frac{64\pi}{3} \text{ cubic units}$$

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