Integration Methods

(a) Standard Integrals and $\int f(ax+b)\,dx$

All the standard derivatives we have learnt provide standard integrals as well.

e.g $\frac{d}{dx}(\tan x) = \sec^2 x$

so $\int \sec^2 x \, dx = \tan x + c$

e.g $\frac{d}{dx}(\ln x) = \frac{1}{x}$

so $\int \frac{1}{x} \, dx = \ln|x| + c$

[NB this is usually written with a modulus sign since we cannot have ln of a negative number.]

Also, we know that $\frac{d}{dx}\left(f(ax+b)\right) = a f'(ax+b)$

e.g $\frac{d}{dx}\left(\sin\left(3x+\frac{\pi}{2}\right)\right) = 3\cos\left(3x+\frac{\pi}{2}\right)$

so $\int 3\cos\left(3x+\frac{\pi}{2}\right) dx = \sin\left(3x+\frac{\pi}{2}\right) + c$

(÷ both sides by 3) $\int \cos\left(3x+\frac{\pi}{2}\right) dx = \frac{1}{3}\sin\left(3x+\frac{\pi}{2}\right) + D$

Generalizing,

$$\boxed{\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + c}$$

Examples

① $\int e^{5x+2}\,dx = \frac{1}{5}e^{5x+2} + c$

② $\int \sec 3x \tan 3x \, dx = \frac{1}{3}\sec 3x + c$

③ $\int \frac{1}{3-2x}\,dx = -\frac{1}{2}\ln|3-2x| + c$

④ $$\int \frac{1}{(3-2x)^4}\,dx = \int (3-2x)^{-4}\,dx$$

$$= -\frac{1}{2} \times -\frac{1}{3}(3-2x)^{-3}$$

$$= \frac{1}{6}(3-2x)^{-3} + c$$

⑤ $$\int \frac{4}{2x-1}\,dx = 4\int \frac{1}{2x-1}\,dx$$

$$= 4 \times \frac{1}{2}\ln|2x-1|$$

$$= 2\ln|2x-1| + c$$

⑥ $$\int \frac{1+\cos x}{\sin^2 x}\,dx = \int \frac{1}{\sin^2 x} + \frac{\cos x}{\sin^2 x}\,dx$$

$$= \int \operatorname{cosec}^2 x + \cot x \operatorname{cosec} x\,dx$$

$$= -\cot x - \operatorname{cosec} x + c$$

p 83 Ex 6A Q 1adfgi, 2abcgj
Ex 6B Q 1, 3abefg,
p37 Ex 4C Q 1ab, 3

(b) Using Long Division

Use for improper fractions

e.g $$I = \int \frac{x^2-5x+3}{x-2}\,dx$$

$$\begin{array}{r l}
 & x - 3 \\
x-2 \,) & x^2 - 5x + 3 \\
 & \underline{x^2 - 2x} \\
 & \quad -3x + 3 \\
 & \quad \underline{-3x + 6} \\
 & \qquad -3
\end{array}$$

so $$I = \int x - 3 - \frac{3}{x-2}\,dx$$

$$= \frac{1}{2}x^2 - 3x - 3\ln|x-2| + c$$

(c) Using Partial Fractions

Example ①

$$I = \int \frac{3x - 1}{x^2 - 6x + 5} \, dx$$

$$= \int \frac{3x - 1}{(x - 1)(x - 5)} \, dx$$

Let $\dfrac{3x - 1}{(x-1)(x-5)} = \dfrac{A}{x-1} + \dfrac{B}{x-5}$

$$A(x-5) + B(x-1) = 3x - 1$$

$x = 5$: $\quad 4B = 14$

$\qquad B = \frac{7}{2}$

$x = 1$: $\quad -4A = 2$

$\qquad A = -\frac{1}{2}$

$$I = \int \frac{7}{2(x-5)} - \frac{1}{2(x-1)} \, dx$$

$$= \frac{1}{2} \int \frac{7}{x-5} - \frac{1}{x-1} \, dx$$

$$= \frac{1}{2} \left[7 \ln|x-5| - \ln|x-1| \right] + C$$

If the highest power on the top is $\geq$ the highest power on the bottom, it is an IMPROPER FRACTION and we need to use long division.

Example 2

$$I = \int \frac{2x^3 - 15x^2 + 31x - 16}{(x-1)(x-5)} \, dx$$

$$
\begin{array}{r}
2x \quad -3 \\
x^2-6x+5 \;\big)\; 2x^3 - 15x^2 + 31x - 16 \\
2x^3 - 12x^2 + 10x \\
\hline
-3x^2 + 21x - 16 \\
-3x^2 + 18x - 15 \\
\hline
3x - 1
\end{array}
$$

$$I = \int 2x - 3 + \frac{3x-1}{x^2-6x+5} \, dx$$

(see eg 0)
$$= \int 2x - 3 + \frac{7}{2(x-5)} - \frac{1}{2(x-1)} \, dx$$

$$= x^2 - 3x + \frac{1}{2}\Big(7\ln|x-5| - \ln|x-1|\Big) + c$$

p92 Ex 6D Q 1aef, 2bc

(d) Integration by Substitution

This is related to the chain rule for differentiation.

The commonest situation where it is useful is where we have :-

$$\boxed{\int f(g(x))\, g'(x)\, dx \quad \text{or} \quad \int \frac{g'(x)}{f(g(x))}\, dx}$$

Examples

① $I = \int \sec^2 x \tan^4 x \, dx$

(Here $g(x) = \tan x$, $f(x) = x^4$, $g'(x) = \sec^2 x$)

Let $t = \tan x \Rightarrow \frac{dt}{dx} = \sec^2 x$

$\Rightarrow dt = \sec^2 x \, dx$

$\Rightarrow dx = \frac{dt}{\sec^2 x}$

So $$I = \int \cancel{\sec^2 x} \; t^4 \; \frac{dt}{\cancel{\sec^2 x}}$$

$$= \frac{1}{5} t^5$$

$$= \underline{\frac{1}{5} \tan^5 x + c}$$

(2) $$I = \int x (x^2 - 3)^5 \, dx$$

[Note that $g(x) = x^2 - 3$ so to fit the pattern perfectly we should have $g'(x) = 2x$ not just x. But a missing constant multiplier does not affect the method.]

Let $t = x^2 - 3 \Rightarrow \frac{dt}{dx} = 2x$

$\Rightarrow \frac{dt}{2x} = dx$

$$I = \int \cancel{x} \; t^5 \; \frac{dt}{2\cancel{x}}$$

$$= \frac{1}{12} t^6$$

$$= \underline{\frac{1}{12} (x^2 - 3)^6 + c}$$

(3) $$I = \int_{x=0}^{x=\pi/2} \frac{\cos x}{\sqrt{1 + \sin x}} \, dx$$

Let $t = 1 + \sin x \quad \frac{dt}{dx} = \cos x$

$dx = \frac{dt}{\cos x}$

Also $x = 0 \Rightarrow t = 1$

$x = \pi/2 \Rightarrow t = 2$

$$I = \int_{t=1}^{t=2} \frac{\cancel{\cos x}}{\sqrt{t}} \frac{dt}{\cancel{\cos x}}$$

$$= \int_1^2 t^{-1/2}\, dt$$

$$= \left[2t^{1/2}\right]_1^2$$

$$= \underline{\underline{2\sqrt{2} - 2}}$$

(4) $$I = \int \frac{x^2}{x^3+1}\, dx$$

Let $t = x^3 + 1 \Rightarrow \frac{dt}{dx} = 3x^2$

$$dx = \frac{dt}{3x^2}$$

$$I = \int \frac{\cancel{x^2}}{t} \frac{dt}{3\cancel{x^2}}$$

$$= \frac{1}{3}\int \frac{1}{t}\, dt$$

$$= \underline{\frac{1}{3}\ln|x^3+1| + c}$$

[or $\int \frac{1}{3t}\, dt = \frac{1}{3}\ln|3t| + c$

$$= \frac{1}{3}\left(\ln 3 + \ln|t|\right) + c$$

$$= \frac{1}{3}\ln|t| + \left(\frac{1}{3}\ln 3 + c\right)$$

$$= \frac{1}{3}\ln|t| + D$$ which is the same as the answer above]

p 94 Ex 6E Q 1 adgh, 2 cdegh
p 99 Ex 6F Q 3 b

(ii) It is possible to use substitution for some integrals which do not fit the above pattern; however, if this is required the substitution to use will be given in the question.

① Use the substitution $t = 9 - 4x$ to evaluate

$$I = \int_{x=0}^{x=2} x\sqrt{9-4x}\, dx$$

Let $t = 9 - 4x \Rightarrow \frac{dt}{dx} = -4$

$$\frac{dt}{-4} = dx$$

Also if $x = 0$, $t = 9$
if $x = 2$, $t = 1$

So $$I = \int_{t=9}^{t=1} x\sqrt{t}\,\frac{dt}{-4}$$

To express the remaining 'x' in terms of t,

$t = 9 - 4x \Rightarrow 4x = 9 - t$

$x = \frac{1}{4}(9-t)$

So $$I = \int_{t=9}^{t=1} \frac{1}{4}(9-t)\,t^{1/2}\,\frac{dt}{-4}$$

$$= -\frac{1}{16}\int_9^1 9t^{1/2} - t^{3/2}\, dt$$

[Note: swapping the limits changes the sign of the answer so we have removed the "–" sign.]

$$= \frac{1}{16}\left[\cancel{9}^{3} \times \frac{2}{\cancel{3}_{1}} t^{3/2} - \frac{2}{5}t^{5/2}\right]_1^9$$

$$= \frac{1}{16}\left[\left(162 - \frac{486}{5}\right) - \left(6 - \frac{2}{5}\right)\right]$$

$$= \underline{\underline{3.7}}$$

p 99 Ex 6F Q 1ac, 2acd, 3bf

(iii) Sometimes the suggested substitution involves t^2 instead of t. In this case we need to use implicit differentiation.

(2) Using the substitution $t^2 = 4 - x$, find

$$\int \frac{3x}{\sqrt{4-x}}\,dx$$

Let $t^2 = 4 - x$

Diff both sides wrt x: $2t\frac{dt}{dx} = -1$

$$\left[\frac{d}{dx}(t^2) = \frac{d}{dt}(t^2)\frac{dt}{dx} = 2t\frac{dt}{dx}.\right]$$

$$2t\,dt = -dx$$
$$dx = -2t\,dt$$

Also $t^2 = 4 - x$
$\Rightarrow x = 4 - t^2$

$$\text{So } I = -\int \frac{3(4-t^2)}{\cancel{\sqrt{t^2}}}\,2\cancel{t}\,dt$$

$$= -\int 24 - 6t^2\,dt$$

$$= \int 6t^2 - 24\,dt$$

$$= 2t^3 - 24t + c$$

(But $t = (4-x)^{1/2}$) $\quad = \underline{\underline{2(4-x)^{3/2} - 24(4-x)^{1/2} + c}}$

(e) Integration by Parts

This is based on the product rule for differentiation.

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx}$$

(Integrate both sides wrt x)

$$uv = \int \frac{du}{dx}v\,dx + \int u\frac{dv}{dx}\,dx$$

$$\boxed{\int u\frac{dv}{dx}\,dx = uv - \int \frac{du}{dx}v\,dx}$$

This replaces one integral with a different (hopefully easier!) integral. It is only useful in certain situations:—

(i) A product, one part of which is linear.

e.g. $I = \int (2x+3)\sin x \, dx$

Let $u = 2x+3$ $\quad \frac{dv}{dx} = \sin x$

$\frac{du}{dx} = 2$ $\quad v = -\cos x$

$$I = -(2x+3)\cos x + \int 2\cos x \, dx$$

$$= -(2x+3)\cos x + 2\sin x + c$$

(ii) A product, one part of which is quadratic (we need to use parts twice for this)

e.g. $I = \int (x^2+3x+4)\cos x \, dx$

Let $u = x^2+3x+4$ $\quad \frac{dv}{dx} = \cos x$

$\frac{du}{dx} = 2x+3$ $\quad v = \sin x$

$$I = (x^2+3x+4)\sin x - \int (2x+3)\sin x \, dx$$

Use parts again as in (i):

$$I = (x^2+3x+4)\sin x - \left[-(2x+3)\cos x + 2\sin x\right] + c$$

(iii) A product of a polynomial and $\ln x$

[In this case let $u = \ln x$ even if the polynomial is linear or quadratic.]

e.g. $$I = \int_1^2 (x^2 + 3x + 4)\ln x \, dx$$

Let $u = \ln x$ $\quad \frac{dv}{dx} = x^2 + 3x + 4$

$\frac{du}{dx} = \frac{1}{x}$ $\quad v = \frac{1}{3}x^3 + \frac{3}{2}x^2 + 4x$

$$I = \left[\ln x\left(\tfrac{1}{3}x^3 + \tfrac{3}{2}x^2 + 4x\right)\right]_1^2 - \int_1^2 \frac{1}{x}\left(\tfrac{1}{3}x^3 + \tfrac{3}{2}x^2 + 4x\right) dx$$

$$= \left[\ln 2\left(\tfrac{50}{3}\right) - 0\right] - \int_1^2 \tfrac{1}{3}x^2 + \tfrac{3}{2}x + 4 \, dx$$

$$= \frac{50\ln 2}{3} - \left[\tfrac{1}{9}x^3 + \tfrac{3}{4}x^2 + 4x\right]_1^2$$

$$= \frac{50\ln 2}{3} - \left[\frac{107}{9} - \frac{175}{36}\right]$$

$$= \underline{\underline{\frac{50\ln 2}{3} - \frac{253}{36}}}$$

(iv) We can prove one more standard integral:—

$$I = \int 1 \ln x \, dx$$

Let $u = \ln x$ $\quad \frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{1}{x}$ $\quad v = x$

$$I = x\ln x - \int \frac{1}{x} \times x \, dx$$

$$\boxed{\int \ln x \, dx = x\ln x - x + c}$$

e.g. $$\int \ln(2x+5)\,dx = \frac{1}{2}\Big[(2x+5)\ln(2x+5) - (2x+5)\Big] + c$$

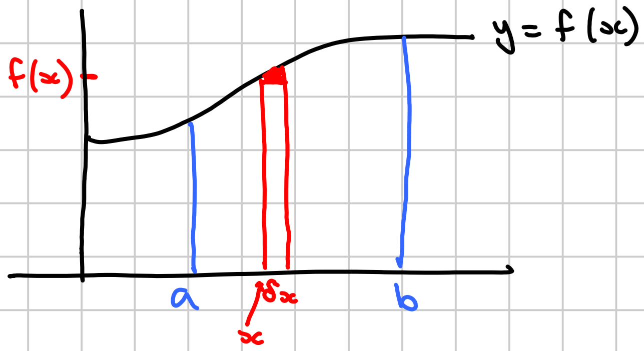
(using the rule from section (a) of notes).

p 102 Ex 6G Q 1 ac, 2 ac, 3 bc, 4 ad

Finding Areas and Volumes using integration

An integral can be thought of as the sum of an infinite number of infinitesimal quantities.

e.g



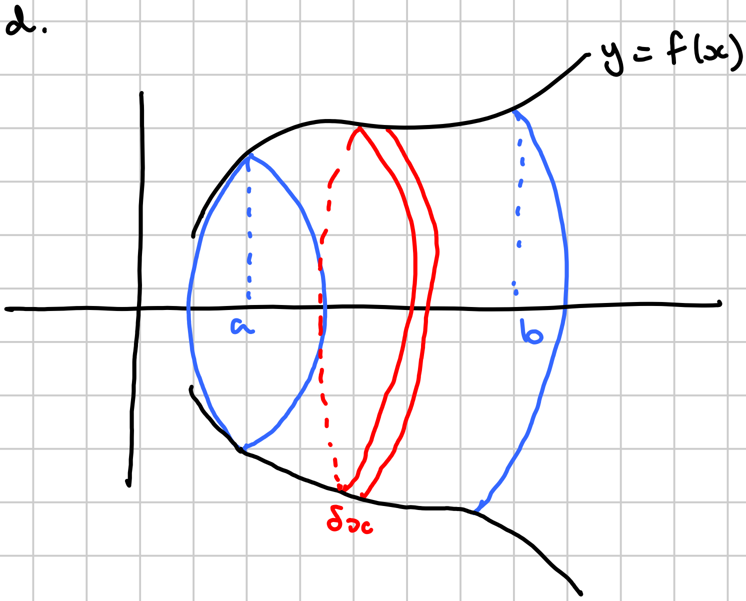
We can think of an area under a curve as being made up of strips, each of which is δx wide.
Each strip is approximately a rectangle $f(x)$ high ($f(x)$ is different for each rectangle).
As the width of each strip becomes infinitesimal, this approximation becomes exact.

So the area is $\displaystyle \lim_{\delta x \to 0} \sum_{x=a}^{x=b} f(x)\,\delta x$

This becomes $\displaystyle \int_{x=a}^{x=b} f(x)\,dx$

Solids of Revolution

If we rotate an area around the x-axis, we form a solid.



We can visualise this solid as made up of discs which are approximately cylinders with "h" $= \delta x$ and "r" $= f(x)$.
So the volume of each cylinder is $\pi[f(x)]^2 \delta x$ and the total volume is

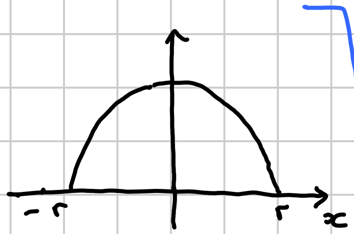
$$\lim_{\delta x \to 0} \left(\sum_{x=a}^{x=b} \pi [f(x)]^2 \delta x \right)$$

$$= \int_a^b \pi [f(x)]^2 \, dx$$

$$\boxed{\text{Volume} = \pi \int_a^b [f(x)]^2 \, dx}$$

<u>Example</u> If the curve $y = \sqrt{r^2 - x^2}$ is rotated through 2π around the x-axis to form a solid, find its volume.

$y = \sqrt{r^2 - x^2}$ can be written as $y^2 = r^2 - x^2$
$x^2 + y^2 = r^2$
which is a circle with radius r
But since $y > 0$ we only have a semicircle.



$$\text{Volume} = \pi \int_{-r}^{r} \left(\sqrt{r^2 - x^2}\right)^2 dx$$

$$= \pi \int_{-r}^{r} r^2 - x^2 \, dx$$

$$= \pi \left[r^2 x - \tfrac{1}{3} x^3 \right]_{-r}^{r}$$

$$= \pi \left[\left(r^3 - \tfrac{1}{3} r^3 \right) - \left(-r^3 + \tfrac{1}{3} r^3 \right) \right]$$

$$= \frac{4}{3} \pi r^3$$ which is the volume of a sphere

p 107 Ex 6I Q 1 abc, 3