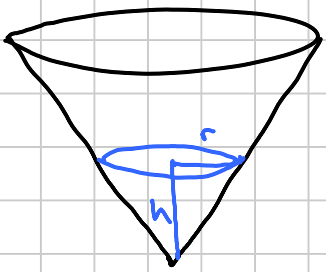
More Differentiation

Related rates of change

Examples

(1) A cone with radius equal to its height is being filled with water at a constant rate of $2\text{cm}^3/\text{s}$. At time t seconds the height of the water in the cone is h cm



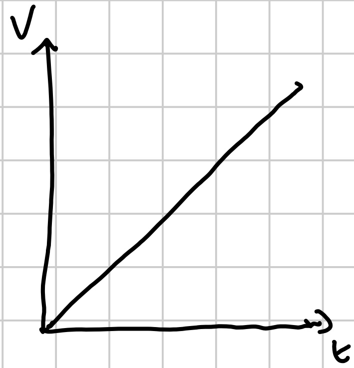
Find the rate at which the height is increasing (i) when $h = 1$
(ii) when $h = 2$

Volume of water is $V = \frac{1}{3}\pi r^2 h$

But since $r = h$, $V = \frac{1}{3}\pi h^3$

There are 3 variables, V, h and t which are all related:

$V = \frac{1}{3}\pi h^3$



$\frac{dV}{dt} = 2$

We need to find $\frac{dh}{dt}$

We know $\frac{dV}{dt} = 2$ and $V = \frac{1}{3}\pi h^3$

$$\Rightarrow \frac{dV}{dh} = \pi h^2$$

So
$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$
$$= \frac{1}{\pi h^2} \times 2 = \frac{2}{\pi h^2}$$

If $h = 1$, $\frac{dh}{dt} = \frac{2}{\pi}$

If $h = 2$, $\frac{dh}{dt} = \frac{2}{4\pi} = \frac{1}{2\pi}$

(2) The volume of a cube is increasing at a constant rate of $6 \text{ cm}^3/\text{s}$.
Find the rate at which the surface area is increasing when the sides of the cube are 2 cm long.

$\frac{dV}{dt} = 6$ We need $\frac{dA}{dt}$. Let side of cube $= x$ at time t

$V = x^3$ $\qquad$ $A = 6x^2$

$\Rightarrow \frac{dV}{dx} = 3x^2$ $\qquad$ $\frac{dA}{dx} = 12x$

$$\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dV} \times \frac{dV}{dt}$$

$$= 12x \times \frac{1}{3x^2} \times 6 = \frac{24}{x}$$

When $x = 2$, $\frac{dA}{dt} = \underline{\underline{12 \text{ cm}^2/\text{s}}}$

Black C3/C4 book p 208 Ex 9.4 Q 1,3,5,7,8,9,12

Differentiating functions defined implicitly

We need to be able to find the gradient of curves defined by equations such as

$$y^3 - x^2 y = 3x + 5$$

To do this we need to be able to differentiate a function of y, with respect to x.

We know that $\frac{d}{dx}(x^3) = 3x^2$

To find $\frac{d}{dx}(y^3)$, we write

$$\frac{d}{dx}(y^3) = \frac{d}{dy}(y^3) \times \frac{dy}{dx} = 3y^2 \frac{dy}{dx}$$

To find $\frac{d}{dx}(\tan y)$, we write

$$\frac{d}{dx}(\tan y) = \frac{d}{dy}(\tan y) \times \frac{dy}{dx} = \sec^2 y \ \frac{dy}{dx}$$

$\boxed{\text{Whenever we need to differentiate a function of } \underline{y} \text{ w.r.t } \underline{x}, \text{ we differentiate w.r.t } y \text{ in the usual way, then multiply by } \frac{dy}{dx}}$

__Examples__

(1) Find the gradient of the curve $y^3 - x^2 y = 3x + 3$ at the point $(1, 2)$.

$$y^3 - x^2 y = 3x + 3$$

Diff both sides with respect to x:

$$3y^2 \frac{dy}{dx} - \underbrace{\left[2xy + x^2 \, 1 \frac{dy}{dx}\right]}_{\text{Product rule}} = 3$$

$$\frac{dy}{dx}\left(3y^2 - x^2\right) = 3 + 2xy$$

$$\frac{dy}{dx} = \frac{3 + 2xy}{3y^2 - x^2}$$

At $(1, 2)$, the gradient is $\frac{3+4}{12-1} = \underline{\underline{\frac{7}{11}}}$

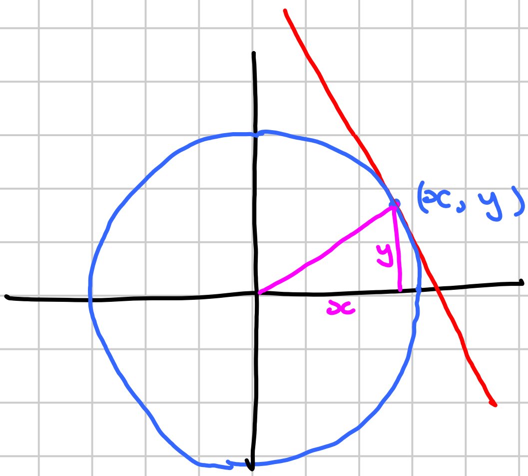
(2) Find the gradient of $x^2 + y^2 = 25$ at the point (x, y)

Diff w.r.t. x

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{x}{y}$$



Gradient of radius $= \frac{y}{x}$

Gradient of tangent $= \frac{-x}{y}$

$\frac{y}{x} \times \frac{-x}{y} = -1 \Rightarrow$ tangent is perpendicular to radius.

(3) Find the gradient of $\frac{e^{x+y}}{y^2} = \frac{1}{4}$ at the point $(2,-2)$.

Get rid of fractions: $4e^{x+y} = y^2$

Method 1:

$$4e^x e^y = y^2$$

$$4e^x e^y + 4e^x e^y \frac{dy}{dx} = 2y\frac{dy}{dx}$$

$$\frac{dy}{dx}\left(4e^x e^y - 2y\right) = -4e^x e^y$$

$$\frac{dy}{dx} = \frac{-4e^x e^y}{4e^x e^y - 2y}$$

$$= \frac{-4e^{x+y}}{4e^{x+y} - 2y}$$

$$= \frac{-2e^{x+y}}{2e^{x+y} - y}$$

At $(2,-2)$, gradient $= \frac{-2}{2-(-2)} = \frac{-1}{2}$

Method 2: $4e^{x+y} = y^2$

(Take ln of both sides) $\ln\left(4e^{x+y}\right) = \ln\left(y^2\right)$

$$\ln 4 + \ln e^{x+y} = 2\ln y.$$

$$\ln 4 + x + y = 2\ln y$$

(Diff w.r.t x) $0 + 1 + 1\frac{dy}{dx} = \frac{2}{y}\frac{dy}{dx}$

$$1 = \frac{dy}{dx}\left(\frac{2}{y} - 1\right)$$

$$\frac{dy}{dx} = \frac{1}{\frac{2}{y} - 1}$$

At $(2,-2)$ gradient $= \frac{1}{-1-1} = \frac{-1}{2}$

p 36 Ex 4B Q 1 acefh, 2, 3, 4

Derivative of $y = a^x$

We can write a as $e^{\ln a}$

So $y = a^x \Rightarrow y = e^{(\ln a)x}$

$\Rightarrow \frac{dy}{dx} = (\ln a)e^{(\ln a)x}$

$$\boxed{y = a^x \Rightarrow \frac{dy}{dx} = a^x \ln a}$$

Example Find the gradient of $y = 2^x$ at the point where $x = -3$

$$\frac{dy}{dx} = 2^x \ln 2$$

If $x = -3$, gradient $= 2^{-3} \ln 2$

$= \frac{1}{8} \ln 2$