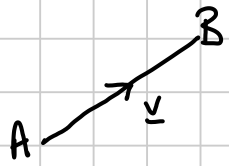
Vectors

Note Title 05/12/2012

(a) Basics

A vector is a quantity which has both MAGNITUDE and DIRECTION. (In C4 the magnitude will be the length.)

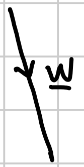
It can be written as $\overrightarrow{AB}$ or $\underline{v}$



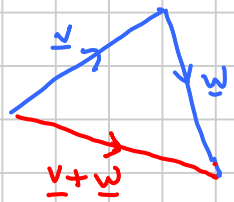
The vector $\overrightarrow{BA}$ is $-\underline{v}$.

To add vectors we place them "nose-to-tail"

e.g



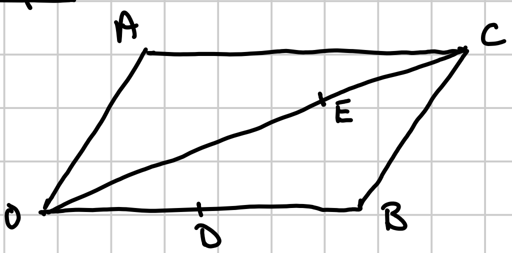
$\underline{v} + \underline{w}$ =



The magnitude of $\underline{v}$ is written $|\underline{v}|$

Note that $\boxed{|\underline{v} + \underline{w}| \neq |\underline{v}| + |\underline{w}|}$

Example



OACB is a parallelogram

D is the midpoint of OB

E divides OC in the ratio 2:1

$\overrightarrow{OA} = \underline{a}$, $\overrightarrow{OB} = \underline{b}$

Find expressions for $\overrightarrow{AD}$ and $\overrightarrow{EB}$ in terms of $\underline{a}$ and $\underline{b}$, and hence state two facts about the lines AD and EB.

$$\overrightarrow{AD} = \overrightarrow{AO} + \tfrac{1}{2}\overrightarrow{OB}$$
$$= -\underline{a} + \tfrac{1}{2}\underline{b}$$

$$\overrightarrow{EB} = \overrightarrow{EO} + \overrightarrow{OB}$$
$$= \tfrac{2}{3}\overrightarrow{CO} + \overrightarrow{OB}$$
$$= -\tfrac{2}{3}(\underline{a}+\underline{b}) + \underline{b}$$
$$= -\tfrac{2}{3}\underline{a} + \tfrac{1}{3}\underline{b}$$

[$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC} = \underline{a} + \underline{b}$

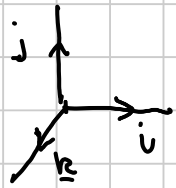
$\overrightarrow{CO} = -(a+b)$]

So $\overrightarrow{EB} = \tfrac{2}{3}\overrightarrow{AD}$ and hence EB is parallel to AD and $2/3$ of the length

[If one vector is a multiple of another, the vectors are parallel.]

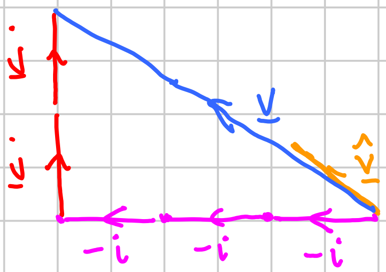
(b) Base vectors

If we define 3 base vectors $\underline{i}$, $\underline{j}$ and $\underline{k}$, each 1 unit long and mutually perpendicular:



then we can define all other vectors in terms of these

e.g.



$\underline{v} = -3\underline{i} + 2\underline{j} + 0\underline{k}$ or $\begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix}$

"column vector"

$$\boxed{\text{If } \underline{v} = a\underline{i} + b\underline{j} + c\underline{k}, \quad |\underline{v}| = \sqrt{a^2 + b^2 + c^2}}$$

A unit vector in the direction of $\underline{v}$ is written $\hat{\underline{v}}$.

so $\boxed{\hat{\underline{v}} = \frac{1}{|\underline{v}|}\underline{v}}$

e.g. if $\underline{v} = -3\underline{i} + 2\underline{j}$, $|\underline{v}| = \sqrt{(-3)^2 + 2^2} = \sqrt{13}$

so $\hat{\underline{v}} = -\frac{3}{\sqrt{13}}\underline{i} + \frac{2}{\sqrt{13}}\underline{j}$ (or $\frac{1}{\sqrt{13}}(-3\underline{i} + 2\underline{j})$)

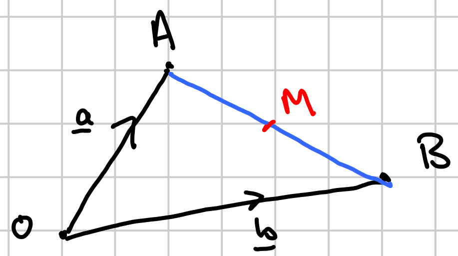
p 50 Ex 5A Q 4
p 53 Ex 5B Q 2 (draw the diagram), 3
p 63 Ex 5F Q 1abc, 2 and e, 4, 5
p 59 Ex 5D Q 3

(c) Position Vectors

A vector does not really represent a point, since it has magnitude and direction.
However vectors are commonly used to refer to points. To do this we choose an origin O, and refer to each point A, B, C etc by the vector $\vec{OA}$, $\vec{OB}$, $\vec{OC}$ etc which is called the POSITION VECTOR (p.v.) of the point, and generally written $\underline{a}$, $\underline{b}$, $\underline{c}$ etc.

So a point A with coordinates $(3, 4, -2)$ will have a p.v. $\underline{a} = 3\underline{i} + 4\underline{j} - 2\underline{k}$ or $\begin{pmatrix} 3 \\ 4 \\ -2 \end{pmatrix}$.

Some results about position vectors:
Let A and B be two points with pvs $\underline{a}$ and $\underline{b}$

① 

$$\vec{AB} = \underline{b} - \underline{a}$$

② If M (with pv $\underline{m}$) is the midpoint of AB

then
$$\begin{aligned} \underline{m} &= \vec{OM} \\ &= \vec{OA} + \tfrac{1}{2}\vec{AB} \\ &= \underline{a} + \tfrac{1}{2}(\underline{b} - \underline{a}) \\ \underline{m} &= \tfrac{1}{2}(\underline{a} + \underline{b}) \end{aligned}$$

(the "average" of $\underline{a}$ and $\underline{b}$)

③ If AB is parallel to CD.

then $\underline{b} - \underline{a} = k(\underline{d} - \underline{c})$

④ If A, B and C are collinear (in a straight line)

then $\underline{b} - \underline{a} = k(\underline{c} - \underline{b})$

p58 Ex 5D Q 2abc, 3 (if not done already)
p63 Ex 5F Q 3, 6

(d) Scalar Product

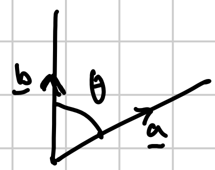
There are two common ways of multiplying vectors but we only need to use one of them.
One is written $\underline{a}.\underline{b}$ and the other $\underline{a} \times \underline{b}$

The SCALAR PRODUCT is written $\underline{a}.\underline{b}$ and so is often called the "DOT PRODUCT".

It is defined as:

$$\boxed{\underline{a}.\underline{b} = |\underline{a}|\,|\underline{b}| \cos\theta}$$

where θ is the angle between the vectors placed 'tail-to-tail'



It follows that:

1. $\underline{a}.\underline{b}$ is a scalar (ie, just a number) not a vector.
2. $\underline{a}.\underline{b} = \underline{b}.\underline{a}$
3. $\underline{a}.(\underline{b}+\underline{c}) = \underline{a}.\underline{b} + \underline{a}.\underline{c}$ (we will not prove this)
4. If $\underline{a}$ and $\underline{b}$ are perpendicular, $\underline{a}.\underline{b} = 0$
   In particular, $\underline{i}.\underline{j} = \underline{j}.\underline{k} = \underline{k}.\underline{i} = 0$
5. If $\underline{a}$ and $\underline{b}$ are parallel (in the same direction),
   $\underline{a}.\underline{b} = |\underline{a}|\,|\underline{b}|$
6. $\underline{a}.\underline{a} = |\underline{a}|^2$
   In particular, $\underline{i}.\underline{i} = \underline{j}.\underline{j} = \underline{k}.\underline{k} = 1$

We can use the above to find a formula for the scalar product of two vectors in $\underline{i}, \underline{j}, \underline{k}$ form.

If $\underline{a} = x_1\underline{i} + y_1\underline{j} + z_1\underline{k}$ and $\underline{b} = x_2\underline{i} + y_2\underline{j} + z_2\underline{k}$
then

$$\underline{a}.\underline{b} = (x_1\underline{i} + y_1\underline{j} + z_1\underline{k}).(x_2\underline{i} + y_2\underline{j} + z_2\underline{k})$$

$$\begin{aligned} = \; & x_1x_2\,\underline{i}.\underline{i} + x_1y_2\,\underline{i}.\underline{j} + x_1z_2\,\underline{i}.\underline{k} \\ & + y_1x_2\,\underline{j}.\underline{i} + y_1y_2\,\underline{j}.\underline{j} + y_1z_2\,\underline{j}.\underline{k} \\ & + z_1x_2\,\underline{k}.\underline{i} + z_1y_2\,\underline{k}.\underline{j} + z_1z_2\,\underline{k}.\underline{k} \end{aligned}$$

$$\boxed{\underline{a}.\underline{b} = x_1 x_2 + y_1 y_2 + z_1 z_2}$$

Examples ① If $\underline{a} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ find the angle between $\underline{a}$ and $\underline{b}$, in degrees to 1 dp.

$$|a| = \sqrt{2^2 + (-2)^2 + 1^2} = 3$$
$$|b| = \sqrt{3^2 + (-1)^2 + 2^2} = \sqrt{14}$$

$$\underline{a}.\underline{b} = 6 + 2 + 2 = 10$$

So $$10 = 3\sqrt{14}\cos\theta$$
$$\cos\theta = \frac{10}{3\sqrt{14}}$$
$$\theta = \underline{\underline{27.0^\circ}}$$

② Find a vector which is perpendicular to both $\underline{a} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} 3 \\ -1 \\ -1 \end{pmatrix}$

Let the perpendicular vector be $\underline{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Then $\underline{a}.\underline{v} = 0 \Rightarrow x + 3y - z = 0$ ①

$\underline{b}.\underline{v} = 0 \Rightarrow 3x - y - z = 0$ ②

The vector $\underline{v}$ is not unique – any multiple of a perpendicular vector will also be perpendicular to $\underline{a}$ and $\underline{b}$.

So we can <u>choose</u> a number for one of the variables.

<u>First</u> eliminate a variable

① − ② $\Rightarrow$ $-2x + 4y = 0$

<u>Now</u> choose a value for y.

Let $y = 1 \Rightarrow x = 2$

<u>Subst</u> these into one of the original equns $\Rightarrow z = 5$

A perpendicular vector is $\begin{pmatrix}2\\1\\5\end{pmatrix}$ (or $\begin{pmatrix}4\\2\\10\end{pmatrix}$ etc)

p69 Ex 5G Q 1, 2ab, 3cdf, 4ace, 5, 7, 8, 9, 12, 13

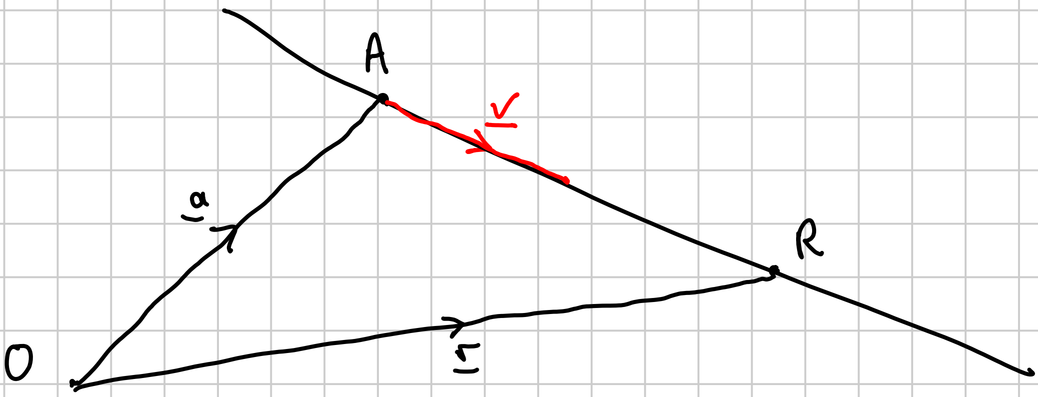
(e) The Vector Equation of a Line

To specify the Cartesian eqn of a line,
$y - y_1 = m(x - x_1)$
we need to know the coords (x_1, y_1) of a point on the line and the gradient m.

Similarly we can specify the vector eqn of a line if we know the position vector (pv) of a point A on the line and a 'direction vector' $\underline{v}$ which points in the direction of the line.

One advantage of vectors is that the same definition works for 2D, 3D, etc.

Let R be a general point on the line.



Now whatever the position of R, we can write
$$\overrightarrow{OR} = \overrightarrow{OA} + \overrightarrow{AR}$$

$$\boxed{\underline{r} = \underline{a} + \lambda \underline{v}}$$

As R moves on the line, λ takes different values

This is the vector eqn of a line. It is not unique - A can be any point on the line, and $\underline{v}$ could be replaced by a multiple of itself.

Examples

① (a) L_1 is the line through A (2, 1, 3) and B (2, 4, 5). Find a vector equation of L_1

A direction vector for the line is $\vec{AB} = \underline{b} - \underline{a}$

$$= \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$$

So the line is $\underline{r} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$

(b) Does the point C (2, 13, 10) lie on L_1?

If so, $\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 13 \\ 10 \end{pmatrix}$ for some value of λ

$$\begin{aligned} 2 + 0\lambda &= 2 && \checkmark \\ 1 + 3\lambda &= 13 && \Rightarrow \lambda = 4 \\ 3 + 2\lambda &= 10 && \Rightarrow \lambda = 7/2 \end{aligned} \Bigg\} \text{ contradiction}$$

So C does not lie on the line.

(c) L_2 is the line $\underline{r} = \begin{pmatrix} 5 \\ 7 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

Do L_1 and L_2 cross?

Note : in 3D lines may be PARALLEL
or they may CROSS
or they may be SKEW

If L_1 and L_2 cross, it will be where

$$\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

This gives 3 equations for two unknowns :—

$$\begin{aligned} 2 &= 5 + \mu & \text{①} \\ 1 + 3\lambda &= 7 + 2\mu & \text{②} \\ 3 + 2\lambda &= 2 + \mu & \text{③} \end{aligned}$$

Solve ① and ② to find λ and μ.

① $\Rightarrow$ $\mu = -3$

② $\Rightarrow$ $1 + 3\lambda = 1$

$\lambda = 0$

Now subst these into ③ $\Rightarrow$ $3 + 0 = 2 - 3$ NO

So the lines do not cross

Also, they are not parallel because $\begin{pmatrix}1\\2\\1\end{pmatrix}$ is not a multiple of $\begin{pmatrix}0\\3\\2\end{pmatrix}$ (these are the direction vectors)

If they did cross, substitute λ (or μ) back into the eqn of its line to find the point of intersection.

(d) Find the angle between the lines L_1 and L_2

This is the angle between their direction vectors.

$$|v_1| = \sqrt{0^2+3^2+2^2} = \sqrt{13}$$

$$|v_2| = \sqrt{1^2+2^2+1^2} = \sqrt{6}$$

$$\underline{v}_1 \cdot \underline{v}_2 = 0 + 6 + 2 = 8$$

$$\theta = \cos^{-1}\left(\frac{8}{\sqrt{13}\sqrt{6}}\right) = 25.1^\circ$$

(If the answer is obtuse, subtract from 180° to get the acute angle between the lines.)

p 73 Ex 5H Q 1(e), 2(a), 3, 4(a), 5(a)

p 75 Ex 5I Q 1, 2, 3

p 77 Ex 5J Q 1, 6

p 77 Ex 5K Q 4, 6, 14