# Coordinate Geometry - Parametric Equations

Note Title 25/01/2013

The equation of a curve is usually given in Cartesian form.

e.g. $y = x^3 - x$ ( explicit Cartesian eqn $y=f(x)$ )

$x^2 + y^2 = 25$ ( implicit " " )

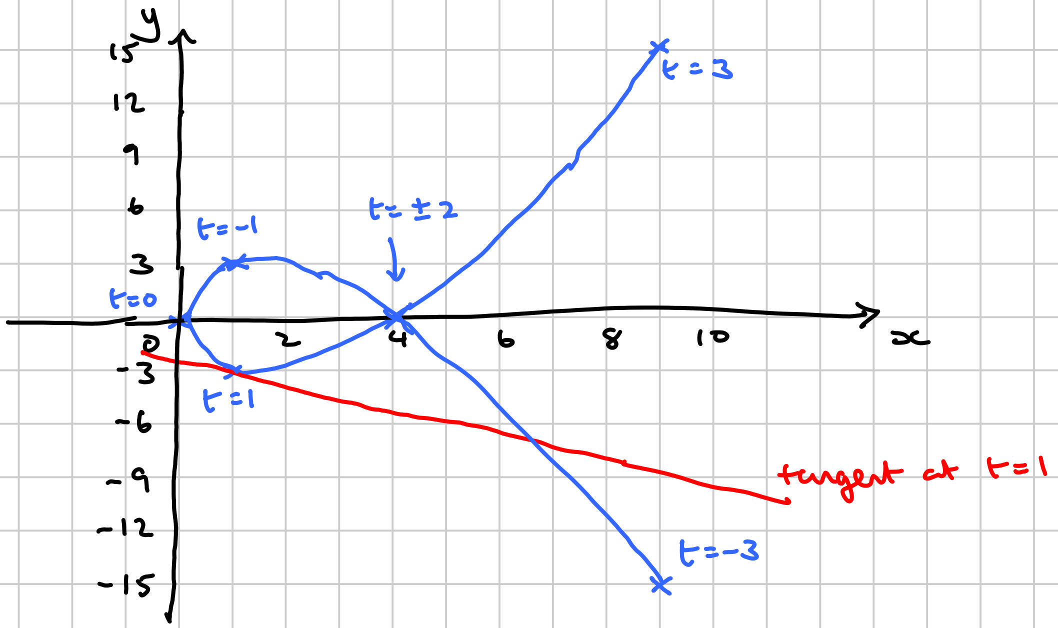
An alternative is to have two equations which define $x$ and $y$ in terms of a third variable, or parameter $t$ or $\theta$.

e.g. $\left.\begin{aligned} x &= t^2 \\ y &= t^3 - 4t \end{aligned}\right\}$

## Examples

① (a) Plot the above curve for $-3 \le t \le 3$

| $t$ | -3 | -2 | -1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $x$ | 9 | 4 | 1 | 0 | 1 | 4 | 9 |
| $y$ | -15 | 0 | 3 | 0 | -3 | 0 | 15 |




(b) Find $\frac{dy}{dx}$ for this curve.

$$x = t^2 \Rightarrow \frac{dx}{dt} = 2t$$

$$y = t^3 - 4t \Rightarrow \frac{dy}{dt} = 3t^2 - 4$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 3t^2 - 4 \times \frac{1}{2t}$$

$$= \frac{3t^2-4}{2t}$$

[ Note that this in terms of $t$ - in most work on parametric equations, we do working in terms of $t$. ]

(c) Find the equation of the tangent to the curve at the point where $t = 1$.

At $t = 1$, $\frac{dy}{dx} = -\frac{1}{2}$ ← To find the normal, we would use the perpendicular gradient ie 2. Apart from that the working is the same

$x = 1$

$y = -3$

( 2 for normal )

So eqn of tangent is $y - (-3) = -\frac{1}{2}(x-1)$

$$y + 3 = -\frac{1}{2}x + \frac{1}{2}$$

$$y = -\frac{1}{2}x - 2\tfrac{1}{2}$$

or $2y = -x - 5$

or $x + 2y + 5 = 0$

(d) Find the coordinates of the point where this line meets the curve again.

Substitute the eqns of the curve into the eqn of line

$$t^2 + 2(t^3 - 4t) + 5 = 0$$

$$2t^3 + t^2 - 8t + 5 = 0$$

One solution of this must be $t = 1$, because we know the line meets the curve at $t = 1$.

So $(t-1)$ must be a factor.

$$\begin{array}{r|l} & 2t^2 + 3t - 5 \\ \hline t-1 & 2t^3 + t^2 - 8t + 5 \\ & 2t^3 - 2t^2 \\ \hline & 3t^2 - 8t \\ & 3t^2 - 3t \\ \hline & -5t + 5 \\ & -5t + 5 \\ \hline & 0 \end{array}$$

$$(t-1)(2t^2+3t-5) = 0$$
$$(t-1)(t-1)(2t+5) = 0$$

So $t = 1$ or $\underline{t = -5/2.}$

[This is a repeated root. This is to be expected since the line touches the curve rather than crossing it.]

Other crossing point is $x = (-5/2)^2 = 25/4$
$y = (-5/2)^3 - 4(-5/2) = -45/8$

(e) Find a Cartesian eqn for this curve.

$$\left.\begin{aligned} x &= t^2 \\ y &= t^3 - 4t \end{aligned}\right\} \quad \text{eliminate } t$$

$$y = t(t^2-4)$$

$$y = \pm\sqrt{x}\,(x-4)$$

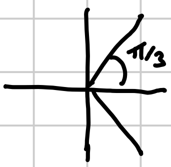
or $\underline{y^2 = x(x-4)^2}$

(2) A curve is defined by equations

$$\left.\begin{aligned} x &= 4\cos t - 2 \\ y &= 4\sin t + 5 \end{aligned}\right\} \quad 0 \leq t \leq 2\pi$$

(a) Find the coordinates of the points where this crosses the y-axis.

At $x = 0$, $4\cos t - 2 = 0$
$\cos t = 1/2$
$t = \pi/3$ or $2\pi - \pi/3 = 5\pi/3$




So $y = 4\sin(\pi/3) + 5 = 2\sqrt{3} + 5$
or $y = 4\sin(5\pi/3) + 5 = 5 - 2\sqrt{3}$

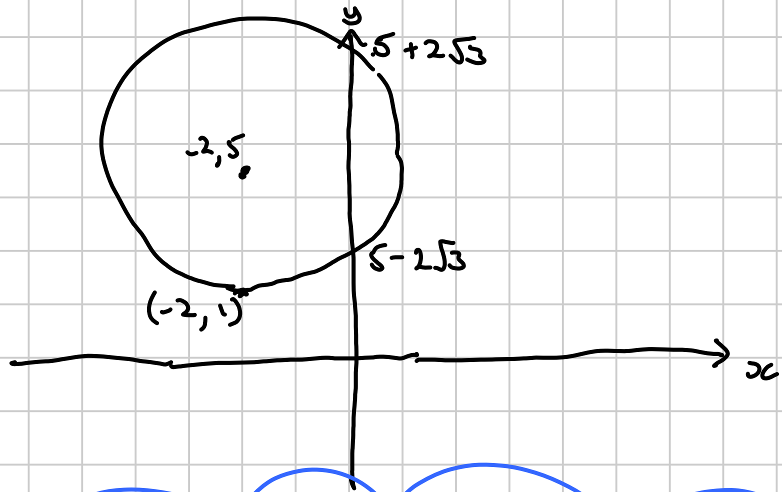
$\underline{(0, 5+2\sqrt{3}) \text{ and } (0, 5-2\sqrt{3})}$

(b) Find a Cartesian equation for this curve and sketch it.

$$x = 4\cos t - 2 \Rightarrow \cos t = \frac{x+2}{4}$$
$$y = 4\sin t + 5 \Rightarrow \sin t = \frac{y-5}{4}$$

$$\cos^2 t + \sin^2 t = 1 \Rightarrow \frac{(x+2)^2}{16} + \frac{(y-5)^2}{16} = 1$$
$$\Rightarrow (x+2)^2 + (y-5)^2 = 16$$

This is a circle, centre $(-2, 5)$ radius 4.




p 162 Ex 7.1 Q 4 acdefhk, 9 ad, 10 ad, 11

p 167 Ex 7.2 Q 2c, 4

p 176 Ex 7.3 Q 1 bde, 2 bc, 3 b, 4, 6

Note: A circle with centre $(a, b)$ and radius $r$ has parametric equations

$$\left.\begin{aligned} x &= r\cos\theta + a \\ y &= r\sin\theta + b \end{aligned}\right\}$$

and Cartesian eqn

$$(x-a)^2 + (y-b)^2 = r^2$$

③ A curve has parametric equations

$$x = \sin\theta$$
$$y = \sin 2\theta$$

Find a Cartesian eqn for this curve.

$$y = 2\sin\theta\cos\theta$$
$$y = 2x\cos\theta$$

Also
$$\cos^2\theta + \sin^2\theta = 1$$
$$\cos^2\theta = 1 - x^2$$
$$\cos\theta = \sqrt{1-x^2}$$

$$\underline{\underline{y = 2x\sqrt{1-x^2}}}$$

④ Find the area enclosed by the loop of the curve
$$x = t^2$$
$$y = t^3 - 4t$$
(see example ①)

Area between curve and $x$-axis

$$= \int_{x=0}^{x=4} y \, dx$$

$$\boxed{\text{We need to replace } dx \text{ by } \frac{dx}{dt}\,dt}$$

$$= \int_{x=0}^{x=4} y \frac{dx}{dt}\,dt$$

$$= \int_{t=0}^{t=-2} (t^3 - 4t)\,2t\,dt$$

$$= \int_{0}^{-2} 2t^4 - 8t^2\,dt$$

$$= \left[\frac{2t^5}{5} - \frac{8t^3}{3}\right]_0^{-2}$$

$$= \left(-\frac{64}{5} - -\frac{64}{3}\right) - (0)$$

$$= \frac{128}{15}$$

Area of whole $= \frac{128}{15} \times 2 = \frac{256}{15}$ sq units

Ex 7.4 Q 1e, 2, 3ab, 4, 5