The Binomial Theorem

In C2 we used the Binomial Theorem for n a positive integer :-

$$\boxed{(a+b)^n = a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + \dots\dots b^n}$$

This is just an identity for multiplying out brackets, and gives a finite number of terms.

The Binomial Theorem for n <u>not</u> a positive integer is a way of finding the power series for a function of the form $(1+x)^n$:-

$$\boxed{(1+x)^n = 1 + nx + \frac{n(n-1)}{2\times1}x^2 + \frac{n(n-1)(n-2)}{3\times2\times1}x^3 + \dots\dots}$$

- This is an INFINITE SERIES
- It is only valid if $|x| < 1$ ie, $-1 < x < 1$ (so that the terms of the series get smaller as the power increases)
- The first term in the bracket must be 1.

Example

① (a) Find the binomial expansion of $\sqrt{1+x}$ as far as the term in x^3.

$$(1+x)^{1/2} = 1 + \tfrac{1}{2}x + \frac{(1/2)(-1/2)}{2}x^2 + \frac{(1/2)(-1/2)(-3/2)}{3\times2}x^3$$

$$= 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 \dots\dots$$

(b) Use your series to find an approximation for $\sqrt{0.9}$

Put $x = -0.1$ $\quad \sqrt{0.9} = 1 + \frac{1}{2}(-0.1) - \frac{1}{8}(-0.1)^2 + \frac{1}{16}(-0.1)^3$

$$= 0.9486875\dots$$

(From calculator, $\sqrt{0.9} = 0.9486832\dots$)

[Note: Although ${}^nC_2 = \frac{n(n-1)}{2\times1}$ (etc), it doesn't make sense in terms of choices to write $\frac{n(n-1)}{2\times1}$ as ${}^nC_2$ if n is not a positive integer.]

② Find the expansion of $\frac{1}{1-2x}$ as far as the term in x^3, and state the interval for which the expansion is valid.

$$(1-2x)^{-1} = 1 + (-1)(-2x) + \frac{(-1)(-2)}{2\times1}(-2x)^2 + \frac{(-1)(-2)(-3)}{3\times2\times1}(-2x)^3$$

$$= 1 + 2x + 4x^2 + 8x^3 + \dots$$

Valid for $|-2x| < 1$

$\Rightarrow |2x| < 1$

$|x| < \frac{1}{2}$ (or $-\frac{1}{2} < x < \frac{1}{2}$)

[By the way, this series is a geometric series:

$$a + ar + ar^2 + ar^3 + \dots$$

where $a = 1$ and $r = 2x$ so $S_\infty = \frac{a}{1-r} = \frac{1}{1-2x}$

which is where we started.]

Ex 3A Q 1 adefh, 3, 4

③ (a) Expand $\sqrt{2+x}$ as far as the term in x^4

This is not a 1! We need to fix this.

So we write: $(2+x)^{1/2} = \left[2\left(1+\frac{1}{2}x\right)\right]^{1/2}$

$= 2^{1/2}\left(1+\frac{1}{2}x\right)^{1/2}$

Now we can use Binomial thm

Remember that $(ab)^n = a^n b^n$

$$= 2^{1/2}\left(1 + \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}x\right) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2\times 1}\left(\tfrac{1}{2}x\right)^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\cancel{\frac{3}{2}})}{\cancel{3}\times 2\times 1}\left(\tfrac{1}{2}x\right)^3 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\cancel{\frac{3}{2}})(-\frac{5}{2})}{4\times\cancel{3}\times 2\times 1}\left(\tfrac{1}{2}x\right)^4\right)$$

$$= \sqrt{2}\left(1 + \tfrac{1}{4}x - \frac{1}{32}x^2 + \frac{1}{128}x^3 - \frac{5}{2048}x^4 + \dots\right)$$

(b) Hence expand $\sqrt{\dfrac{2+x}{1-x^2}}$ as far as the term in x^4

This is $(2+x)^{1/2}(1-x^2)^{-1/2}$

Now $(1-x^2)^{-1/2} = 1 + \left(-\tfrac{1}{2}\right)(-x^2) + \dfrac{(-\frac{1}{2})(-\frac{3}{2})}{2\times 1}(-x^2)^2$

$$= 1 + \tfrac{1}{2}x^2 + \tfrac{3}{8}x^4$$

So $(2+x)^{1/2}(1-x^2)^{-1/2}$

"x^4" terms

$$= \sqrt{2}\left(1 + \tfrac{1}{4}x - \tfrac{1}{32}x^2 + \tfrac{1}{128}x^3 - \tfrac{5}{2048}x^4\right)\left(1 + \tfrac{1}{2}x^2 + \tfrac{3}{8}x^4\right)$$

$$= \sqrt{2}\left(1 + \tfrac{1}{4}x + \left(\tfrac{1}{2} - \tfrac{1}{32}\right)x^2 + \left(\tfrac{1}{8} + \tfrac{1}{128}\right)x^3 + \left(\tfrac{3}{8} - \tfrac{5}{2048} - \tfrac{1}{64}\right)x^4\right)$$

$$= \sqrt{2}\left(1 + \tfrac{1}{4}x + \frac{15}{32}x^2 + \frac{17}{128}x^3 + \frac{731}{2048}x^4 + \dots\right)$$

(c) State the interval for which this expansion is valid

The series for $(1+\tfrac{1}{2}x)^{1/2}$ is valid for $|\tfrac{1}{2}x| < 1$

$\Rightarrow |x| < 2$

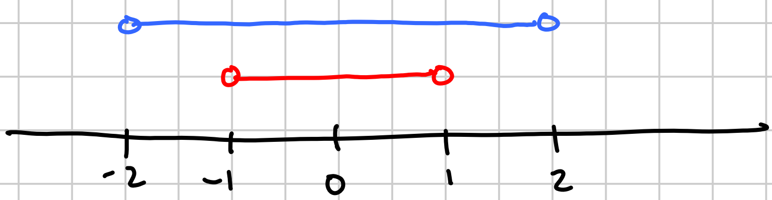
ie, $-2 < x < 2$

The series for $(1-x^2)^{-1/2}$ is valid for $|-x^2| < 1$

$\Rightarrow |x^2| < 1$

$|x| < 1$

ie, $-1 < x < 1$



The overall series is only valid if <u>both</u> these are valid

ie, for $-1 < x < 1$

p28 Ex 3B Q 1 abcegh, 3

p30 Ex 3C Q 1

p31 Ex 3D Q 1h