Partial Fractions

Identities and Equations

An <u>identity</u> is true for all values of x

e.g. $(2x-1)(x+3) \equiv 2x^2+5x-3$

$x=1$ $\quad 1 \times 4 = 2+5-3$ ✓

$x=6$ $\quad 11 \times 9 = 72+30-3$ ✓

An <u>equation</u> is true for a limited number of values of x.

e.g.
$$\begin{aligned}(2x-1)(x+3) &= 2x^2+x+7\\ 2x^2+5x-3 &= 2x^2+x+7\\ 5x-3 &= x+7\\ 4x &= 10\\ x &= 2.5\end{aligned}$$

check: $4 \times 5.5 = 12.5+2.5+7$ ✓

but any other number will not work.

Polynomials

A polynomial is an expression

$$ax^n + bx^{n-1} + \dots + cx^2 + dx + e$$

The highest power of x is the <u>degree</u> of the polynomial.

degree 1 : linear
degree 2 : quadratic
degree 3 : cubic
degree 0 : constant

A polynomial fraction is <u>proper</u> if the degree of the top is less than the degree of the bottom - otherwise it is <u>improper</u>.

An improper fraction can be converted into a polynomial plus a proper fraction by long division.

Partial Fractions

The process of 'undoing' an addition or subtraction of fractions is called splitting into partial fractions.

Examples

① $\dfrac{7x+6}{(x+1)(x+2)}$

We want to write this as $\dfrac{A}{x+1} + \dfrac{B}{x+2}$

Work this out : $\dfrac{A(x+2)+B(x+1)}{(x+1)(x+2)}$

Compare numerators : $7x+6 \equiv A(x+2)+B(x+1)$

Two methods for finding A and B:

Method 1 : Substituting values for x

$x=-1 \qquad -1 = A$

$x=-2 \qquad -8 = -B \quad \Rightarrow \quad B=8$

Method 2 : comparing coefficients.

"x" s : $\quad 7 = A + B \quad$ ①

constant : $\quad 6 = 2A + B \quad$ ②

② − ① $\quad -1 = A$

subst. $\quad 7 = -1 + B \quad \Rightarrow \quad B = 8$

So $\quad \dfrac{7x+6}{(x+1)(x+2)} \equiv \dfrac{8}{x+2} - \dfrac{1}{x+1}$

(2) $$\frac{4}{(x^2+1)(x+1)}$$

Write this as $$\frac{Bx+C}{x^2+1} + \frac{A}{x+1}$$

$$= \frac{(Bx+C)(x+1) + A(x^2+1)}{(x^2+1)(x+1)}$$

Compare numerators: $4 \equiv (Bx+C)(x+1) + A(x^2+1)$

Subst $x=-1$: $4 = 2A$
$A = 2$

Subst $x=0$: $4 = C + 2$
$C = 2$

Subst $x=1$: $4 = (B+2)2 + 4$
$0 = 2(B+2)$
$B = -2$

So $$\frac{4}{(x^2+1)(x+1)} = \frac{-2x+2}{x^2+1} + \frac{2}{x+1}$$

(3) $$\frac{3x-1}{(x+3)(x-2)^2}$$ ← Repeated bracket.

This cannot be written $\frac{A}{x+3} + \frac{B}{x-2} + \frac{C}{x-2}$

The required form is

$$\frac{A}{x+3} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$= \frac{A(x-2)^2 + B(x-2)(x+3) + C(x+3)}{(x+3)(x-2)^2}$$

Compare numerators

Subst $x=2$: $5 = 5C$
$C = 1$

Subst $x = -3$ $\quad -10 = 25A$

$$A = -\frac{10}{25} = -\frac{2}{5}$$

Subst $x = 0$

$$-1 = 4A - 6B + 3C$$
$$-1 = -\frac{8}{5} - 6B + 3$$
$$6B = -\frac{8}{5} + 3 + 1$$
$$= \frac{12}{5}$$
$$B = \frac{2}{5}$$

So
$$\frac{3x-1}{(x+3)(x-2)^2} = \frac{-2/5}{x+3} + \frac{2/5}{x-2} + \frac{1}{(x-2)^2}$$
$$= \frac{2}{5(x-2)} + \frac{1}{(x-2)^2} - \frac{2}{5(x+3)}$$

(4) Write $\dfrac{x^3}{(x+2)(x+1)}$ in the form

$$Ax + B + \frac{C}{x+2} + \frac{D}{x+1}$$

This is improper, so first do a long division

$$\begin{array}{r|l}
 & x - 3 \\
x^2 + 3x + 2 & x^3 + 0x^2 + 0x + 0 \\
 & \underline{x^3 + 3x^2 + 2x} \\
 & -3x^2 - 2x + 0 \\
 & \underline{-3x^2 - 9x - 6} \\
 & 7x + 6
\end{array}$$

So
$$\frac{x^3}{(x+2)(x+1)} = x - 3 + \frac{7x+6}{(x+2)(x+1)}$$

Now use partial fractions on the remainder (see eg (1))

$$= \underline{x - 3 + \frac{8}{x+2} - \frac{1}{x+1}}$$

Ex 1B Q 1abgh.

Ex 1D Q 2, 3, 5, 7

Ex 1E Q 1ac, 3, [2b]

Ex 1F Q 1c, 3