Note Title 12/10/2012

Differentiation

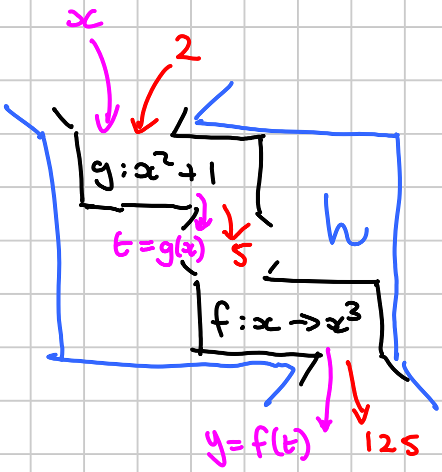
Derivative of sin x and cos x

$$\boxed{\begin{aligned} \frac{d}{dx}(\sin x) &= \cos x \\ \frac{d}{dx}(\cos x) &= -\sin x \end{aligned}}$$

remember this − sign

PROVIDED x IS IN RADIANS

Differentiating a Composite Function



Example If $g(x) = x^2 + 1$
and $f(x) = x^3$

and $h(x) = f(g(x))$

then $h(x) = (x^2+1)^3$

Now suppose $y = h(x)$ and we want to find $\frac{dy}{dx}$

We let t be the result of the first or "inner" function

Let $t = x^2 + 1$ so $y = t^3$

$\frac{dt}{dx} = 2x$ $\qquad$ $\frac{dy}{dt} = 3t^2$

Now we can use the CHAIN RULE to find $\frac{dy}{dx}$

$$\boxed{\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}}$$

$$= 3t^2 \times 2x$$
$$= 3(x^2+1)^2 \times 2x$$
$$= \underline{\underline{6x(x^2+1)^2}}$$

A particular case of the chain rule is

$$\frac{dy}{dx} \times \frac{dx}{dy} = 1$$

$$\boxed{\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}}$$

We can use this to find the derivative of INVERSE functions.

Derivatives of e^x and $\ln x$

We know that $\boxed{\frac{d}{dx}(e^x) = e^x}$ (this is the point of the number e)

Now if $y = \ln x$

then $x = e^y$ (inverse function)

so $\frac{dx}{dy} = e^y$

so $\frac{dy}{dx} = \frac{1}{e^y}$

so in terms of x, $\frac{dy}{dx} = \frac{1}{x}$

ie, $\boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$

More examples

1. $y = e^{\cos x}$

Let $t = \cos x$ so $y = e^t$

$\frac{dt}{dx} = -\sin x$ $\qquad \frac{dy}{dt} = e^t$

$\Rightarrow \frac{dy}{dx} = -\sin x \, e^t$

$= \underline{\underline{-\sin x \, e^{\cos x}}}$

(2) $y = \dfrac{5}{\sqrt{1+\sin^2 x}}$

ie/ $y = 5(1+\sin^2 x)^{-1/2}$

Let $t = \sin x$ and $u = 1 + t^2$ so $y = 5u^{-1/2}$

$\dfrac{dt}{dx} = \cos x$ $\quad \dfrac{du}{dt} = 2t$ $\quad \dfrac{dy}{du} = -\dfrac{5}{2}u^{-3/2}$

So $\dfrac{dy}{dx} = \dfrac{\cancel{dt}}{dx} \times \dfrac{\cancel{du}}{\cancel{dt}} \times \dfrac{dy}{\cancel{du}}$

$= \cos x \times \cancel{2}\sin x \times -\dfrac{5}{\cancel{2}}(1+t^2)^{-3/2}$

$= \underline{\underline{-5\cos x \sin x (1+\sin^2 x)^{-3/2}}}$

(3) $y = \ln\sqrt{4+x^2}$ (first use laws of logs to simplify)

$= \ln(4+x^2)^{1/2}$

$= \frac{1}{2}\ln(4+x^2)$

Let $t = 4 + x^2$ so $y = \frac{1}{2}\ln t$

$\dfrac{dt}{dx} = 2x$ $\quad \dfrac{dy}{dt} = \dfrac{1}{2} \times \dfrac{1}{t} = \dfrac{1}{2t}$

So $\dfrac{dy}{dx} = \cancel{2}x \times \dfrac{1}{\cancel{2}t}$

$= \underline{\underline{\dfrac{x}{4+x^2}}}$

Sheet Q 1 abdfgjklmnqrstwx, 2abc, 3ef, 5, 6

not cehiop uv

Finish for HWK

A function of a linear function

Suppose $\quad y = f(ax+b)$

Let $\quad t = ax+b \qquad$ so $\quad y = f(t)$

$$\frac{dt}{dx} = a \qquad \frac{dy}{dt} = f'(t)$$

So $\quad \dfrac{dy}{dx} = af'(t)$

$$\boxed{\frac{dy}{dx} = af'(ax+b)}$$

We often use this rule to write down a derivative without showing the chain rule working.

e.g

① $y = \sin 5x \Rightarrow \dfrac{dy}{dx} = 5\cos 5x$

② $y = e^{3x+4} \Rightarrow \dfrac{dy}{dx} = 3e^{3x+4}$

③ $y = \ln(2x-1) \Rightarrow \dfrac{dy}{dx} = \dfrac{2}{2x-1}$

④ $y = \ln(kx) \Rightarrow \dfrac{dy}{dx} = \dfrac{k}{kx} = \dfrac{1}{x}$

[Note : this is $y = \ln x + \ln k$
so $\dfrac{dy}{dx} = \dfrac{1}{x} + 0$]

p126 Ex 8D Q 1abce
p128 Ex 8E Q 1acdefg
p130 Ex 8F Q 1abd
p131 Ex 8G Q 1cdegh

The Product Rule

If $\quad y = uv \quad$ (where $u = f(x)$ and $v = g(x)$)

then $\quad \dfrac{dy}{dx} = \dfrac{du}{dx}v + u\dfrac{dv}{dx} \quad \left(\text{or } \dfrac{dy}{dx} = f'(x)g(x) + f(x)g'(x) \right)$

Examples

① $\quad y = e^{x}\sin x$

$$\frac{dy}{dx} = e^{x}\sin x + e^{x}\cos x$$

② $\quad y = \cos^2 x \ln(3-2x)$

$u = \cos^2 x \qquad\qquad v = \ln(3-2x)$

Let $t = \cos x$ so $u = t^2$

$\dfrac{dt}{dx} = -\sin x \qquad \dfrac{du}{dt} = 2t \qquad\qquad \dfrac{dv}{dx} = \dfrac{-2}{3-2x}$

$\dfrac{du}{dx} = -2\sin x\cos x$

So $\quad \dfrac{dy}{dx} = -2\sin x\cos x\ln(3-2x) - \dfrac{2\cos^2 x}{3-2x}$

Ex 4.4 Q 1 abdeg, 2, 4, 6, 7

More standard derivatives

$$y = \tan x = \frac{\sin x}{\cos x}$$

So $$\frac{dy}{dx} = \frac{\cos x \cos x - \sin x \times (-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

($y = \cot x$ can be done the same way)

$$y = \sec x = \frac{1}{\cos x} = (\cos x)^{-1}$$

Let $t = \cos x$ so $y = t^{-1}$

$\frac{dt}{dx} = -\sin x$ $\qquad$ $\frac{dy}{dt} = -t^{-2}$

So $$\frac{dy}{dx} = -\sin x \times -(\cos x)^{-2}$$

$$= \frac{\sin x}{\cos^2 x} = \tan x \sec x$$

Trig derivatives

$f(x)$	$f'(x)$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\tan x \sec x$
$\operatorname{cosec} x$	$-\cot x \operatorname{cosec} x$

Derivative of a^x

$$y = a^x \Rightarrow y = e^{(\ln a)x}$$

$$\Rightarrow \frac{dy}{dx} = (\ln a) e^{(\ln a)x}$$

so $$\boxed{y = a^x \Rightarrow \frac{dy}{dx} = (\ln a) a^x}$$

p134 Ex 8J Q 1a-f, 2cdfk, 3bcdfhkl