Note Title 19/09/2012

Trigonometry

The inverse trigonometric functions can be written as

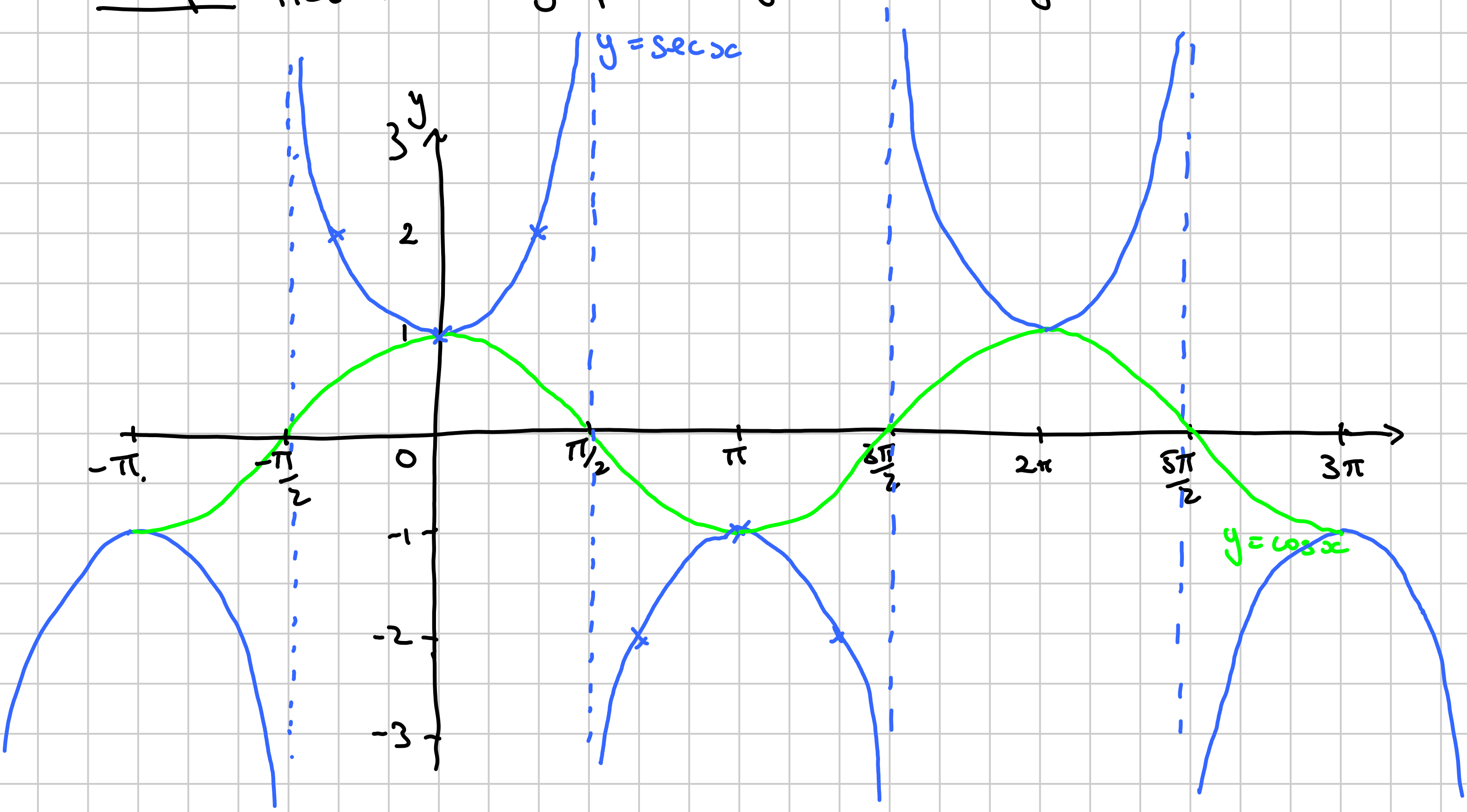
$$y = \sin^{-1} x \quad \text{or} \quad y = \arcsin x \iff x = \sin y$$

and similarly for cos and tan.

The reciprocal trig functions are written as

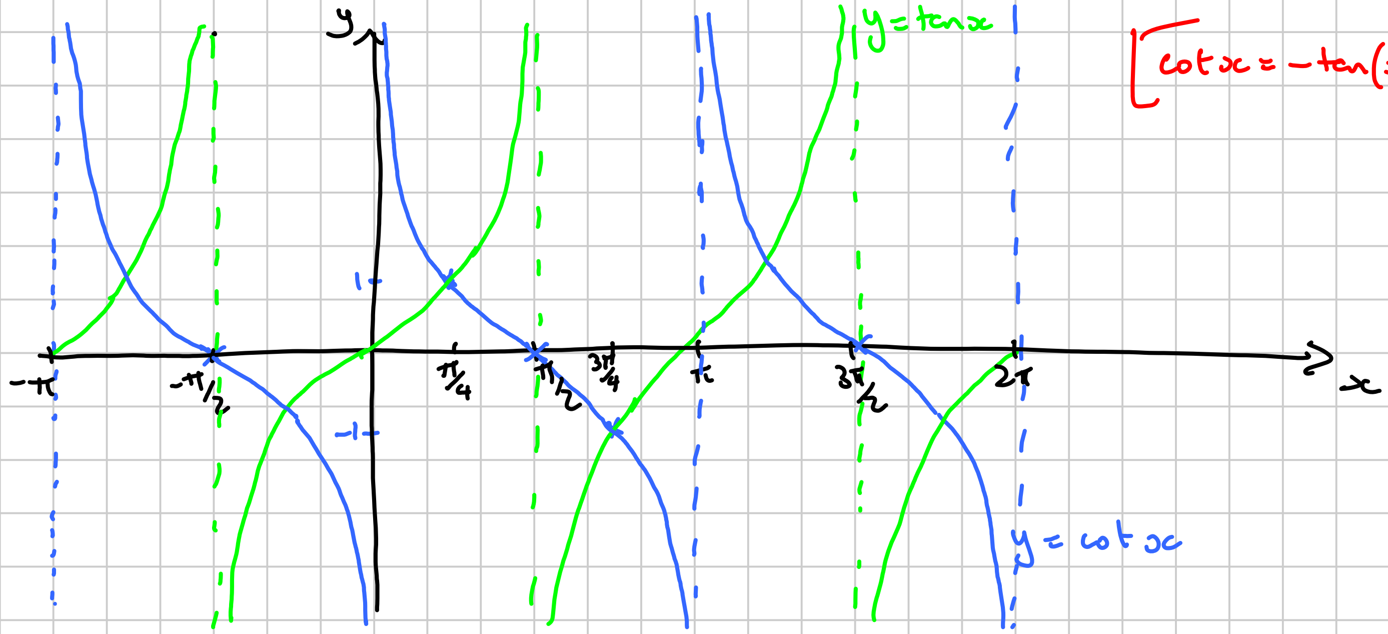
$$\boxed{\begin{array}{lll} \dfrac{1}{\cos x} = \sec x & & (\text{secant}) \\[2ex] \dfrac{1}{\sin x} = \operatorname{cosec} x & & (\text{cosecant}) \\[2ex] \dfrac{1}{\tan x} = \cot x & & (\text{cotangent}) \end{array}}$$

Graphs Here is the graph of $y = \cos x$ and $y = \sec x$



The graph of $\sin x$ is the same as $\cos x$, but translated $\pi/2$ to the right. Likewise for $\operatorname{cosec} x$ and $\sec x$.

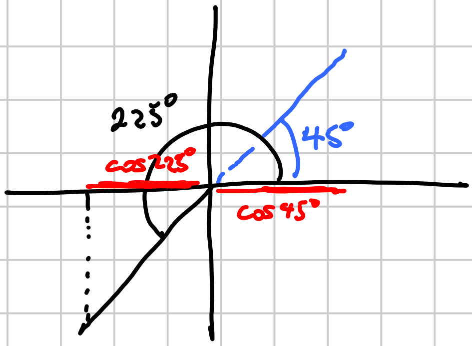
ie, $\cos(x - \pi/2) = \sin x \qquad \sec(x - \pi/2) = \operatorname{cosec} x$



$\boxed{\cot x = -\tan\left(x - \frac{\pi}{2}\right)}$

Example

$$\sec 225^\circ = \frac{1}{\cos 225^\circ} = \frac{1}{-\cos 45^\circ} = \underline{\underline{-\sqrt{2}.}}$$

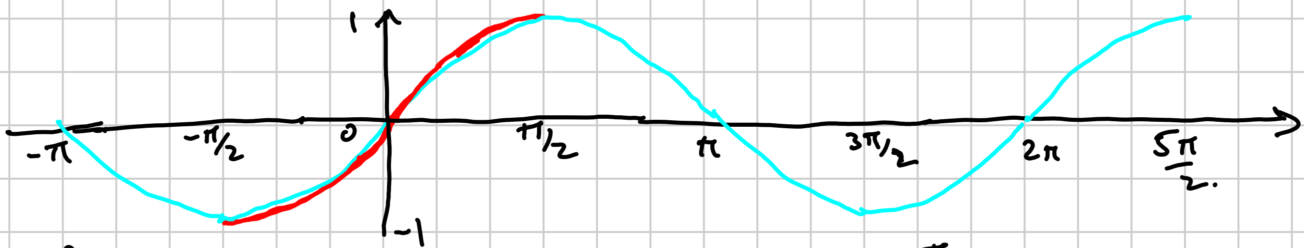


$\cos 45^\circ = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

Ex 6A Q 1, 2adg, 3 (try not to use calculator!)

Inverse Trig Functions

Because sine is a many-to-one function (e.g $\sin\frac{\pi}{6} = \frac{1}{2}$, $\sin\frac{5\pi}{6} = \frac{1}{2}$ etc), in order to define an inverse we have to restrict its domain



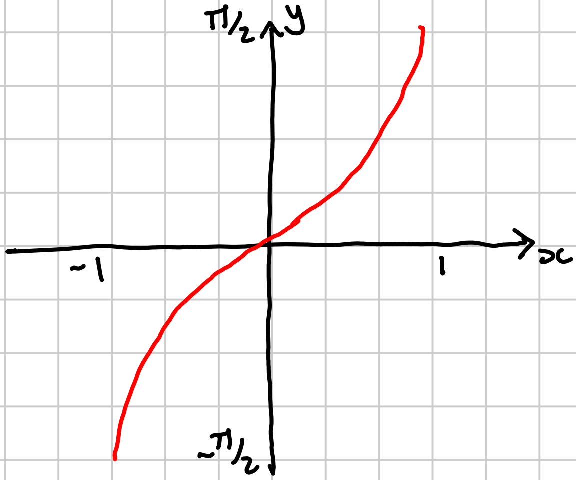
If we make the domain $-\pi/2 \le x \le \pi/2$, $\sin x$ is one-to-one.

Now we can define its inverse, $\arcsin x$

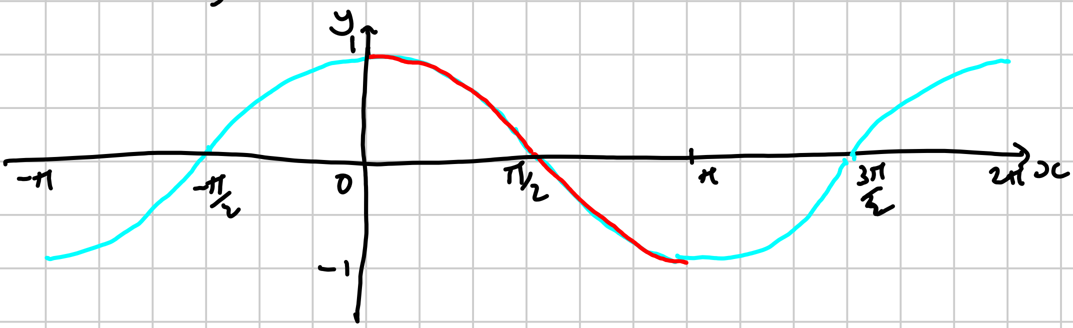
with domain $[-1, 1]$ and range $[-\pi/2, \pi/2]$

$-1 \le x \le 1$ $\qquad -\pi/2 \le \arcsin x \le \pi/2$

Graph of $\arcsin x$

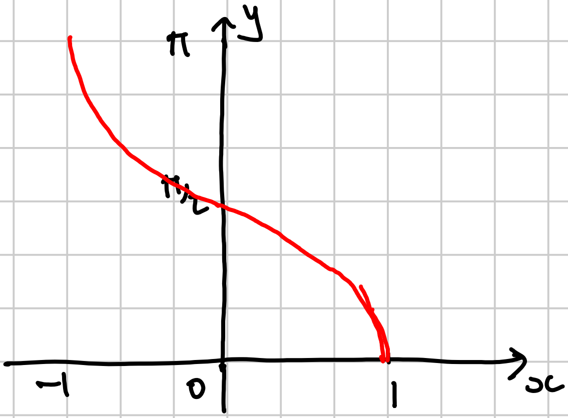


For $\cos x$,



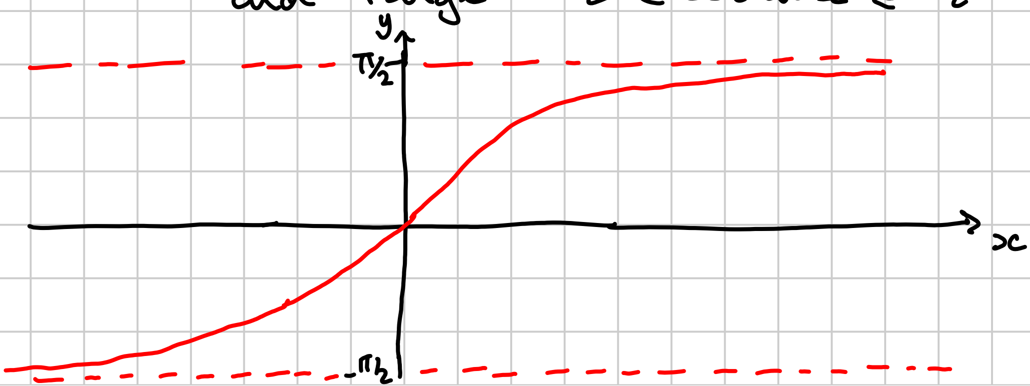
we restrict the domain to $0 \le x \le \pi$

So $\arccos x$ has domain $-1 \le x \le 1$ and range $0 \le \arccos x \le \pi$.



For $\tan x$, we can take the section of graph between the asymptotes at $-\pi/2$ and $\pi/2$.

So $\arctan x$ has domain $\mathbb{R}$ (all real numbers) and range $-\pi/2 \le \arctan x \le \pi/2$



p90 Ex 6E Q 1, 4, 7

Some more identities

Because $\tan\theta = \frac{\sin\theta}{\cos\theta}$, and $\cot\theta = \frac{1}{\tan\theta}$, $\boxed{\cot\theta = \frac{\cos\theta}{\sin\theta}}$

Also, starting from $\cos^2\theta + \sin^2\theta = 1$

(Divide both sides by $\cos^2\theta$) $\quad 1 + \tan^2\theta = \sec^2\theta$

(Divide both sides by $\sin^2\theta$) $\quad \cot^2\theta + 1 = \operatorname{cosec}^2\theta$

Examples

① Prove that $\sec^2\theta - \operatorname{cosec}^2\theta = \tan^2\theta - \cot^2\theta$

$$\text{LHS} = (1 + \tan^2\theta) - (\cot^2\theta + 1)$$
$$= \text{RHS}$$

② Prove that $\cot\theta + \tan\theta = \operatorname{cosec}\theta\sec\theta$

(If in doubt, change everything into sin and cos θ)

$$\text{LHS} \equiv \frac{\cos\theta}{\sin\theta} + \frac{\sin\theta}{\cos\theta}$$
$$= \frac{\cos^2\theta + \sin^2\theta}{\sin\theta\cos\theta}$$
$$= \frac{1}{\sin\theta\cos\theta}$$
$$= \operatorname{cosec}\theta\sec\theta = \text{RHS}$$

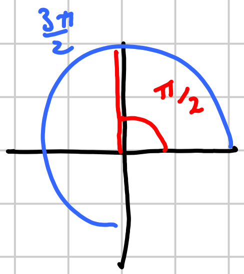
③ Solve the equation $\cot\theta + 2\cos\theta = 0$ for $0 \le \theta \le 2\pi$.

$$\frac{\cos\theta}{\sin\theta} + 2\cos\theta = 0$$

$$\cos\theta + 2\cos\theta\sin\theta = 0$$
$$\cos\theta(1 + 2\sin\theta) = 0$$

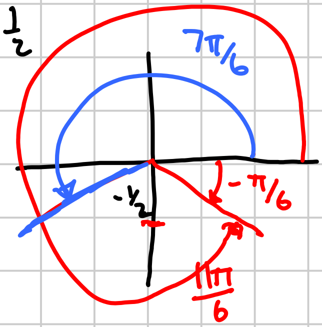
$\cos\theta = 0$ or $1 + 2\sin\theta = 0$

$\sin\theta = -\frac{1}{2}$



$\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$

$\theta = \frac{7\pi}{6}$ or $\frac{11\pi}{6}$



p81 Ex 6C Q 3a-f, 4a-f, 5acefh, 6gh
p85 Ex 6D Q 6bcdef, 8bcg

Odd and Even Functions

If $f(-x) = f(x)$ for all values of x, f is called an EVEN FUNCTION.

The graph of an even function has a line of symmetry on the y-axis.

$\cos(x)$ and $\sec(x)$ are EVEN functions ie, $\boxed{\cos(-\theta) = \cos\theta}$

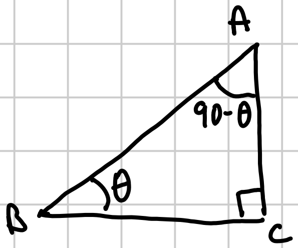
If $f(-x) = -f(x)$ for all values of x, f is called an ODD FUNCTION

The graph of an odd function has half-turn symmetry about the origin.

$\sin(x)$, $\tan(x)$, $\operatorname{cosec} x$ and $\cot(x)$ are all ODD functions

ie, $\boxed{\begin{aligned}\sin(-\theta) &= -\sin(\theta)\\ \tan(-\theta) &= -\tan(\theta)\end{aligned}}$

More common identities :-



$$\boxed{\begin{aligned} \sin(90-\theta) &= \frac{BC}{AB} = \cos\theta \\ \cos(90-\theta) &= \frac{AC}{AB} = \sin\theta \end{aligned}}$$

Compound Angle Formulae

Note that $\sin(A+B) \neq \sin A + \sin B$. !

In fact $\boxed{\sin(A+B) = \sin A\cos B + \cos A\sin B.}$

(we will just assume this)

From this we can obtain the other compound angle formulae :-

Replace B with (-B) : $\sin(A+(-B)) = \sin A\cos(-B) + \cos A\sin(-B)$

$$\boxed{\sin(A-B) = \sin A\cos B - \cos A\sin B}$$

Now $\cos(A+B) = \sin(90-(A+B))$

$= \sin((90-A) - B)$

$= \sin(90-A)\cos B - \cos(90-A)\sin B$

$$\boxed{\cos(A+B) = \cos A\cos B - \sin A\sin B}$$

Replace B with (- B)

$$\cos(A+(-B)) = \cos A\cos(-B) - \sin A\sin(-B)$$

$$\boxed{\cos(A-B) = \cos A\cos B + \sin A\sin B}$$

Now $\tan(A+B) = \dfrac{\sin(A+B)}{\cos(A+B)}$

$$= \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$$

(Divide top and bottom by $\cos A\cos B$)

$$\boxed{\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}}$$

Similarly,

$$\boxed{\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}}$$

Examples

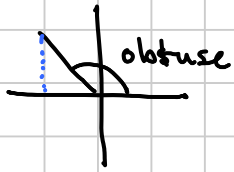
(1) Given that A is obtuse and $\sin A = \frac{3}{5}$

and B is reflex and $\tan B = \frac{5}{12}$

find (a) $\sin(A+B)$ (b) $\cot(A-B)$

$$\cos^2 A = 1 - \sin^2 A$$
$$= 1 - \frac{9}{25}$$
$$= \frac{16}{25}$$
$$\cos A = \pm\frac{4}{5}$$

but A is obtuse



so $\cos A = -\frac{4}{5}$

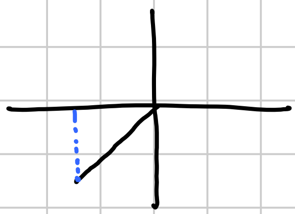
and $\tan A = \dfrac{\sin A}{\cos A} = \dfrac{3/5}{-4/5} = \dfrac{3}{5} \times -\dfrac{5}{4}$

$= -3/4$

$$\sec^2 B = 1 + \tan^2 B$$
$$= 1 + \frac{25}{144}$$
$$= \frac{169}{144}$$
$$\sec B = \pm\frac{13}{12}$$
$$\cos B = \pm\frac{12}{13}$$

but B is reflex and $\tan B$ is +ve

so $\cos B = -12/13$



And $\sin B = \tan B\cos B$

$= \frac{5}{12} \times -\frac{12}{13} = -5/13$

(a) $\sin(A+B) = \sin A\cos B + \cos A\sin B$

$= \frac{3}{5} \times \frac{-12}{13} + \frac{-4}{5} \times \frac{-5}{13}$

$= \frac{-36}{65} + \frac{20}{65}$

$= -16/65$

(b) $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

$= \frac{-3/4 - 5/12}{1 + (-3/4)(5/12)}$

$= \frac{-14/12}{1 - 5/16} = \frac{-7}{\cancel{6}\,3} \times \frac{\cancel{16}\,8}{11}$

$= \frac{-56}{33}$

$\cot(A-B) = -\frac{33}{56}$

(2) Prove that $\frac{\cos(A+B)}{\sin B \cos B} = \frac{\cos A}{\sin B} - \frac{\sin A}{\cos B}$

LHS $= \frac{\cos A\cos B - \sin A \sin B}{\sin B \cos B} = \frac{\cos A \cancel{\cos B}}{\sin B \cancel{\cos B}} - \frac{\sin A \cancel{\sin B}}{\cancel{\sin B} \cos B}$

$=$ RHS

p99 Ex 7A Q 4acdeh, 7ach, 10, 13e, 18

Double Angle Identities

By putting $B = A$ into the identities above, we find

$\sin 2A = 2\sin A\cos A$

$\cos 2A = \cos^2 A - \sin^2 A$

(replace $\cos^2 A$ by $1-\sin^2 A$) $= 1 - 2\sin^2 A$

(replace $\sin^2 A$ by $1-\cos^2 A$) $= 2\cos^2 A - 1$

$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$

Example Find solutions of $\cos 4\theta + \cos 2\theta = 0$ for $0 \le \theta \le \pi$

Since $\cos 4\theta = 2\cos^2 2\theta - 1$,

$$2\cos^2 2\theta + \cos 2\theta - 1 = 0$$

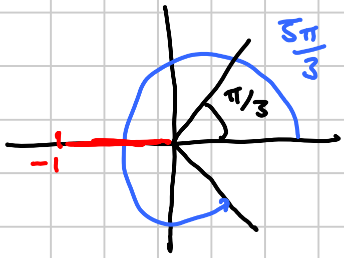
Let $t = \cos 2\theta \Rightarrow$

$$2t^2 + t - 1 = 0$$
$$(2t - 1)(t+1) = 0$$
$$t = \tfrac{1}{2} \text{ or } t = -1$$

So $\cos 2\theta = \frac{1}{2}$ (let $\alpha = 2\theta \Rightarrow 0 \le \alpha \le 2\pi$)

$\cos \alpha = \tfrac{1}{2}$

$\alpha = \pi/3$



$\alpha = \pi/3$ or $5\pi/3$.

Or $\cos \alpha = -1$

$\alpha = \pi$

So $\theta = \frac{1}{2}\alpha = \pi/6$ or $5\pi/6$ or $\pi/2$

p103 Ex 7B Q 1a-d, 3abce, 8

p106 Ex 7C Q. 1acd, 3acde, 5abc, 6, 12

[Solution to 7B Q8

8 (a) (i) $\cos 2A = 2\cos^2 A - 1 = 2 \times \frac{1}{9} - 1 = -\frac{7}{9}$

(ii) $\sin^2 A = 1 - \cos^2 A = \frac{8}{9} \Rightarrow \sin A = \frac{\sqrt{8}}{3} = \frac{2\sqrt{2}}{3}$

(iii) $\sin 2A = 2\sin A \cos A = 2 \times \frac{2\sqrt{2}}{3} \times -\frac{1}{3} = -\frac{4\sqrt{2}}{9}$

$\Rightarrow \operatorname{cosec} 2A = \frac{-9}{4\sqrt{2}}$.

(b) $\tan 2A = \frac{\sin 2A}{\cos 2A} = \frac{-\frac{4\sqrt{2}}{9}}{-\frac{7}{9}} = -\frac{4\sqrt{2}}{9} \times -\frac{9}{7}$

$= \frac{4\sqrt{2}}{7}$ QED]

Sums to Products (and Products to Sums)

From $\sin(A+B) = \sin A\cos B + \cos A\sin B$
and $\sin(A-B) = \sin A\cos B - \cos A\sin B$

We get by adding $\sin(A+B)+\sin(A-B) = 2\sin A\cos B$ ①
or by subtracting $\sin(A+B)-\sin(A-B) = 2\cos A\sin B$ ②

From $\cos(A+B) = \cos A\cos B - \sin A\sin B$
and $\cos(A-B) = \cos A\cos B + \sin A\sin B$

We get by adding $\cos(A+B)+\cos(A-B) = 2\cos A\cos B$ ③
or by subtracting $\cos(A+B)-\cos(A-B) = -2\sin A\sin B$ ④

We want to replace $A+B$ and $A-B$ by single letters

Let $\left.\begin{array}{l} P = A+B \\ Q = A-B \end{array}\right\} \Rightarrow \begin{array}{l} P+Q = 2A \\ P-Q = 2B \end{array}$ $\quad A = \frac{P+Q}{2}$, $B = \frac{P-Q}{2}$

Substituting these into ① – ④ we get the identities for converting sums (or differences) to products

$$\boxed{\begin{aligned} \sin P + \sin Q &= 2\sin\frac{P+Q}{2}\cos\frac{P-Q}{2} \\ \sin P - \sin Q &= 2\cos\frac{P+Q}{2}\sin\frac{P-Q}{2} \\ \cos P + \cos Q &= 2\cos\frac{P+Q}{2}\cos\frac{P-Q}{2} \\ \cos P - \cos Q &= -2\sin\frac{P+Q}{2}\sin\frac{P-Q}{2} \end{aligned}}$$

(In formula book)

<u>Example</u> $\sin 8\theta - \sin 2\theta = 2\cos\frac{8\theta+2\theta}{2}\sin\frac{8\theta-2\theta}{2} = 2\cos 5\theta\sin 3\theta$

- We can also rearrange ① – ④ to convert products into sums

e.g. from ①, ③, ④

$$\boxed{\begin{aligned} \sin A\cos B &= \tfrac{1}{2}\left(\sin(A+B)+\sin(A-B)\right) \\ \cos A\cos B &= \tfrac{1}{2}\left(\cos(A+B)+\cos(A-B)\right) \\ \sin A\sin B &= -\tfrac{1}{2}\left(\cos(A+B)-\cos(A-B)\right) \end{aligned}}$$

<u>Example</u> $\sin 8\theta\cos 2\theta = \frac{1}{2}(\sin 10\theta + \sin 6\theta)$

"$t = \tan\theta/2$" identities

We have seen that $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$.

We can also show that $\sin 2A = \dfrac{2\tan A}{1+\tan^2 A}$

[$$\begin{aligned} \text{RHS} &\equiv \frac{2\tan A}{\sec^2 A} \\ &\equiv \frac{2\sin A}{\cos A} \times \cos^2 A \\ &\equiv 2\sin A\cos A \\ &\equiv \text{LHS} \end{aligned}$$]

and that $\cos 2A = \dfrac{1-\tan^2 A}{1+\tan^2 A}$

[$$\begin{aligned} \text{RHS} &\equiv \frac{1-\dfrac{\sin^2 A}{\cos^2 A}}{\sec^2 A} \\ &= \frac{\cos^2 A - \sin^2 A}{\cos^2 A} \times \cos^2 A \\ &= \cos^2 A - \sin^2 A \\ &= \text{LHS} \end{aligned}$$]

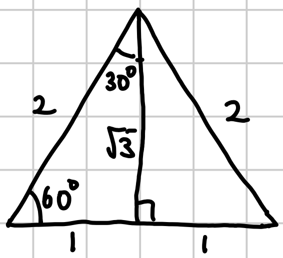
Now by letting $\theta = 2A$, so that $A = \theta/2$
and letting $\tan\theta/2 = t$

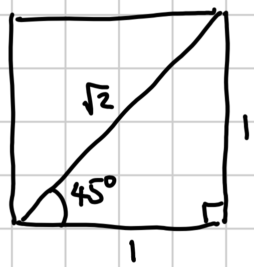
we get

$$\boxed{\begin{aligned} \tan\theta &= \frac{2t}{1-t^2} \\ \sin\theta &= \frac{2t}{1+t^2} \\ \cos\theta &= \frac{1-t^2}{1+t^2}. \end{aligned}}$$

These are useful for converting trig equations into algebraic equations.

Sin, Cos, Tan of Common Angles





Degrees	0°	30°	45°	60°	90°
Radians	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
sin	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1
cos	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0
tan	0	$1/\sqrt{3}$	1	$\sqrt{3}$	∞

Obtuse angles:

$\sin\theta = \sin(180-\theta)$

$\cos\theta = -\cos(180-\theta)$

Examples

① Find the exact value of $\sin\frac{\pi}{8}\cos\frac{\pi}{8}$.

$$\sin\frac{\pi}{8}\cos\frac{\pi}{8} = \frac{1}{2}\left[\sin\left(\frac{\pi}{8}+\frac{\pi}{8}\right) + \sin\left(\frac{\pi}{8}-\frac{\pi}{8}\right)\right]$$

$$= \frac{1}{2}\left[\sin\frac{\pi}{4} + \sin 0\right]$$

$$= \frac{1}{2\sqrt{2}} \quad \times\frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{\sqrt{2}}{4}$$

② Prove that $\sin 13° + \sin 47° = \sin 73°$

$$\text{LHS} = 2\sin 30°\cos(-17°)$$

$$= 2\times\frac{1}{2}\times\cos 17° \quad \text{(since cos is an even function)}$$

$$= \cos 17°$$

$$= \sin(90-17)$$

$$= \sin 73°$$

(This is an example of $\sin\theta + \sin(60-\theta) = \sin(60+\theta)$)

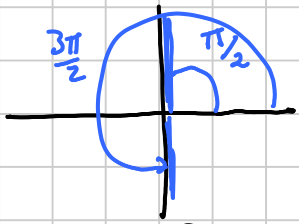
③ Find solutions of $\cos 4\theta + \cos 2\theta = 0$ for $0 \le \theta \le \pi$

$$2\cos 3\theta \cos\theta = 0$$

Either $\cos 3\theta = 0$ or $\cos\theta = 0$

$0 \le \theta \le \pi$

$\Rightarrow 0 \le 3\theta \le 3\pi$
(α)



$\theta = \pi/2$
(already got this solution).

$$\alpha = \pi/2 \text{ or } 3\pi/2 \text{ or } 5\pi/2$$
$$\theta = \pi/6 \text{ or } \pi/2 \text{ or } 5\pi/6$$

p 115 Ex 7E Q 1 cde, 3 bd, 6, 8, 9, 10 a d

The function $a\cos x + b\sin x$

This can be converted into the form $R\sin(x+\alpha)$ (or similar), which shows its properties more clearly.

Example

(a) Express $f(x) = 3\cos x + 4\sin x$ in the form $R\cos(x-\alpha)$

$$R\cos(x-\alpha) = R[\cos x\cos\alpha + \sin x\sin\alpha]$$
$$= R\cos x\cos\alpha + R\sin x\sin\alpha.$$

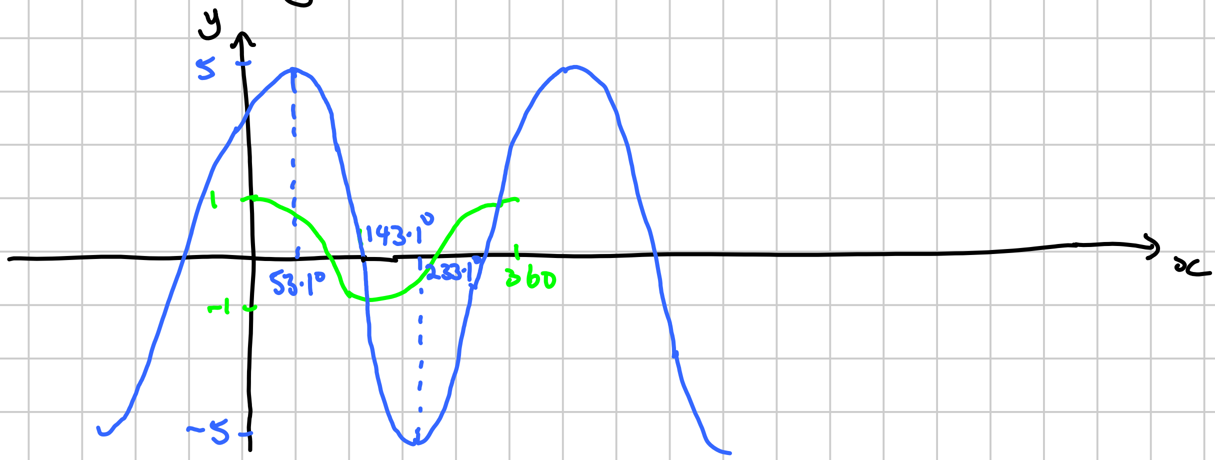
We want this to equal $3\cos x + 4\sin x$

Comparing gives
$$R\cos\alpha = 3 \quad ①$$
$$R\sin\alpha = 4 \quad ②$$

$\frac{②}{①} \Rightarrow \tan\alpha = 4/3$
$$\alpha = \tan^{-1}(4/3)$$
$$= 53.1^\circ$$

$$① ^2 + ② ^2 \Rightarrow R^2\cos^2\alpha + R^2\sin^2\alpha = 3^2 + 4^2$$
$$\Rightarrow R^2(1) = 25$$
$$\Rightarrow R = 5$$

(b) Sketch the graph of $y = 3\cos x + 4\sin x$
This is $y = 5\cos(x - 53.1°)$

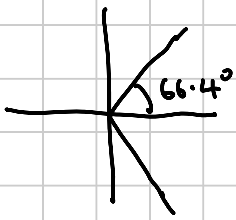


(c) Solve the equation $3\cos x + 4\sin x = 2$
for $0 \le x \le 360°$

$$5\cos(x - 53.1°) = 2$$

$\alpha = x - 53.1°$

$$\cos\alpha = \frac{2}{5}$$
$$\alpha = 66.4°$$



$\alpha = 66.4°$ or $293.6°$

$x = 119.5°$ or $346.7°$

p111 Ex 7D Q 6ad, 7, 8, 9, 10, 12bd, [15]