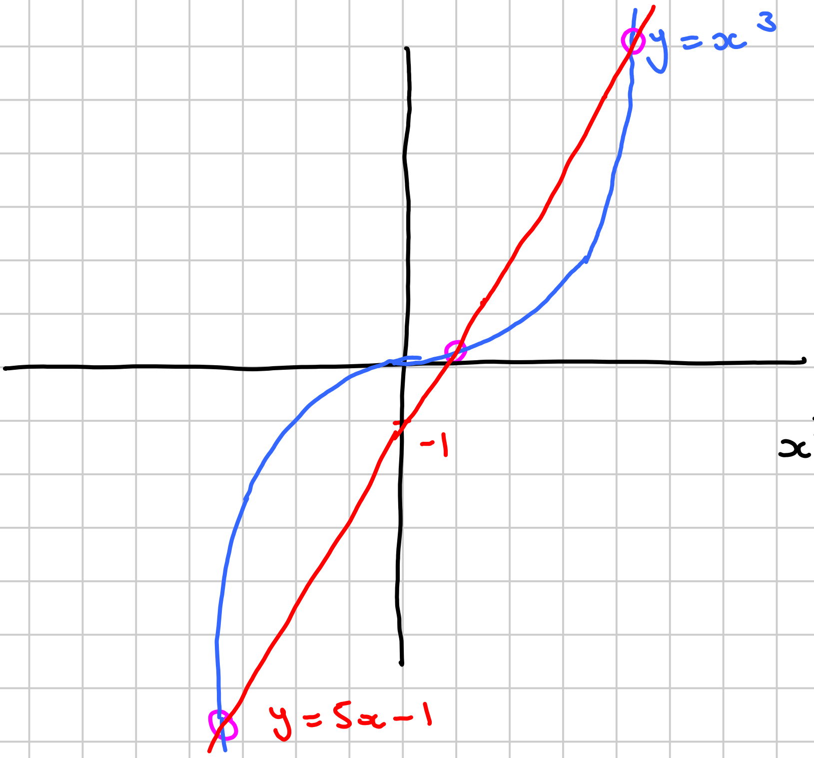
Solving equations using Numerical Methods

Most equations cannot be solved exactly. So we need methods of finding approximate solutions.

- To find how many solutions there are, a sketch graph is useful. It may be easier to rearrange the equation and draw two graphs to see where they cross.

e.g. $x^3 - 5x + 1 = 0$

rewrite as $x^3 = 5x - 1$

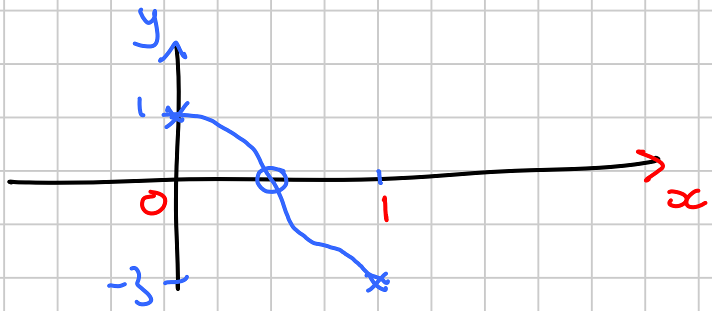


$x^3 - 5x + 1 = 0$ has 3 solutions

- To show that a solution lies in a certain interval we can look for a SIGN CHANGE. To do this we need the equation in the form $f(x) = 0$

e.g if $f(x) = x^3 - 5x + 1$

$$\left.\begin{aligned} f(0) &= 1 \\ f(1) &= -3 \end{aligned}\right\}$$ SIGN CHANGE $\Rightarrow$ there a solution of $x^3 - 5x + 1 = 0$ in the interval $[0, 1]$.



- To improve the accuracy of a solution we can use the '$x = g(x)$' method.

To do this we rearrange the equation so that the LHS is just x and the RHS is a function of x.

e.g. $$x^3 - 5x + 1 = 0$$
$$x^3 = 5x - 1$$
$$x = \sqrt[3]{5x - 1}$$

We now make this into a rule for generating a sequence :—

$$x_{n+1} = \sqrt[3]{5x_n - 1}$$

We then start the sequence with a value for x_0 close to a solution of the equation.

$$x_0 = 0.5$$
$$x_1 = 1.14\ldots$$
$$x_2 = 1.68\ldots$$
$$x_3 = 1.95\ldots$$
$$x_4 = 2.059\ldots$$
$$x_5 = 2.102\ldots$$
$$x_6 = 2.1189\ldots$$
$$x_7 = 2.1249\ldots$$
$$x_8 = 2.1271\ldots$$
$$x_9 = 2.1279\ldots$$

We can see that the sequence is converging to 2.13 to 2 decimal places. This is therefore a solution of the original equation.

- We can check that this is accurate to 2dp by putting the Lower Bound and Upper Bound into the original equation.

LB $x = 2.125$ $\quad x^3 - 5x + 1 = -0.029\ldots$

UB $x = 2.135$ $\quad x^3 - 5x + 1 = 0.056\ldots$

SIGN CHANGE $\Rightarrow$ solution lies between 2.125 and 2.135

$\Rightarrow$ it is 2.13 to 2 dp

[Note that there is often more than one way to rearrange an equation into the form $x = g(x)$.

For example we could write

$$x^3 - 5x + 1 = 0$$

as

$$x^3 + 1 = 5x$$

$$x = \frac{x^3+1}{5}$$

giving an iterative formula $x_{n+1} = \dfrac{x_n^3+1}{5}$.

Starting with $x_0 = 0.5$ as before, we get

$x_1 = 0.225$ $\quad x_2 = 0.2022\ldots$ $\quad x_3 = 0.20165\ldots$

$x_4 = 0.20164$ and $x_5, x_6 \ldots$ are the same to 5sf.

Note that

- Different rearrangements may converge to different roots (or may diverge).
- Some rearrangements converge more quickly than others.]

p44 Ex 4A Q 3, 4, 6, 7, 8, 9.

p 50 Ex 4B Q 1, 5, 6, 7, 8