# Exponential and Logarithmic Functions

For any exponential function $y = a^x$,
the derivative is $\frac{dy}{dx} = k \times a^x$.

We can find a number such that $k = 1$ ie
the derivative is IDENTICAL to the original function.

This number is called $e$, and $e = 2.71828\ldots$
(an irrational number like $\pi$).

$$\boxed{\begin{array}{l} \text{So if } y = e^x, \text{ then } \frac{dy}{dx} = e^x \\ \text{It follows that } \int e^x \, dx = e^x + c \end{array}}$$

Example The number of people with a disease $t$ years after it was first detected is given by

$$N = 300 - 100e^{-0.5t}$$

(a) How many people were first diagnosed?

If $t = 0$, $N = 300 - 100e^0$
$= \underline{\underline{200}}$

(b) How many people have the disease 2 years after it was detected?

If $t = 2$, $N = 300 - 100e^{-1}$
$= \underline{\underline{263}}$

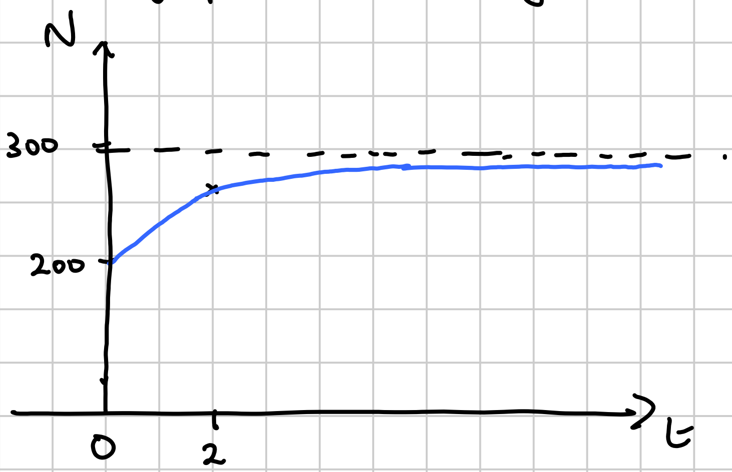
(c) What is the expected number of people with the disease in the long term?

As $t \to \infty$, $e^{-0.5t} \to 0$

$$\text{so} \quad N \to 300 - 100 \times 0$$
$$N \to 300$$

The number of people with the disease will increase towards 300 but not past this number.

(d) Sketch the graph of $N$ against $t$ (for $t \geqslant 0$)




Ex 3A Q 1 abcdf, 2, 3, 5

## The Logarithmic Function

We know that if $y = 2^x$, then $x = \log_2 y$

Similarly, the inverse function of $y = e^x$

is $x = \log_e y$

We have a shorthand for $\log_e$ : it is often written ln.
So all the usual rules of logs apply to ln :—

$$\boxed{\begin{aligned} \ln a + \ln b &= \ln(ab) \\ \ln a - \ln b &= \ln\left(\frac{a}{b}\right) \\ \ln e &= 1 \\ k \ln a &= \ln(a^k) \end{aligned}}$$

Because $e^x$ and $\ln x$ are inverse functions,

$$\boxed{\begin{aligned} \ln(e^{x}) &= x \\ e^{\ln x} &= x \end{aligned}}$$

Example (continued)

Given that $N = 300 - 100e^{-0.5t}$

($N$ = no of people with the disease, $t$ = no of years since disease started)

(e) After how many years will the number of cases reach 280?

$$280 = 300 - 100e^{-0.5t}$$

$$100e^{-0.5t} = 300 - 280 = 20$$

$$e^{-0.5t} = \frac{20}{100} = 0.2$$

$$\ln(e^{-0.5t}) = \ln(0.2)$$

$$-0.5t \cancel{\ln e} = \ln(0.2)$$

$$t = \frac{\ln(0.2)}{-0.5}$$

$$= \underline{\underline{3.22 \text{ years}}}$$

(f) Write a formula for $t$ in terms of $N$.

$$N = 300 - 100e^{-0.5t}$$

$$100e^{-0.5t} = 300 - N$$

$$e^{-0.5t} = \frac{300-N}{100}$$

$$\ln(e^{-0.5t}) = \ln\left(\frac{300-N}{100}\right)$$

$$-0.5t(\cancel{\ln e}) = \ln\left(\frac{300-N}{100}\right)$$

$$(\times -2) \qquad (\times -2)$$

$$t = -2\ln\left(\frac{300-N}{100}\right)$$

**Example 2** Solve the follow equations giving exact solutions

(a)

$$3\ln(x-2) = 12$$

$(\div 3)$ $(\div 3)$

$$\ln(x-2) = 4$$

$$e^{\ln(x-2)} = e^4$$

$$x - 2 = e^4$$ (note that $e^{\ln a} = a$)

$$x = e^4 + 2$$

(b)

$$3 + 2e^{4x} = 6$$

$$2e^{4x} = 3$$

$$e^{4x} = \frac{3}{2}$$

$$\ln(e^{4x}) = \ln\left(\frac{3}{2}\right)$$

$$4x = \ln\left(\frac{3}{2}\right)$$

$$x = \frac{1}{4}\ln\left(\frac{3}{2}\right)$$

**Example 3** Solve $\ln(x+1) - \ln 2 = 3$

[ CANT do $e^{\ln(x+1)} - e^{\ln 2} = e^3$
- can only do this with the whole of each side ]

$$\ln\left(\frac{x+1}{2}\right) = 3$$

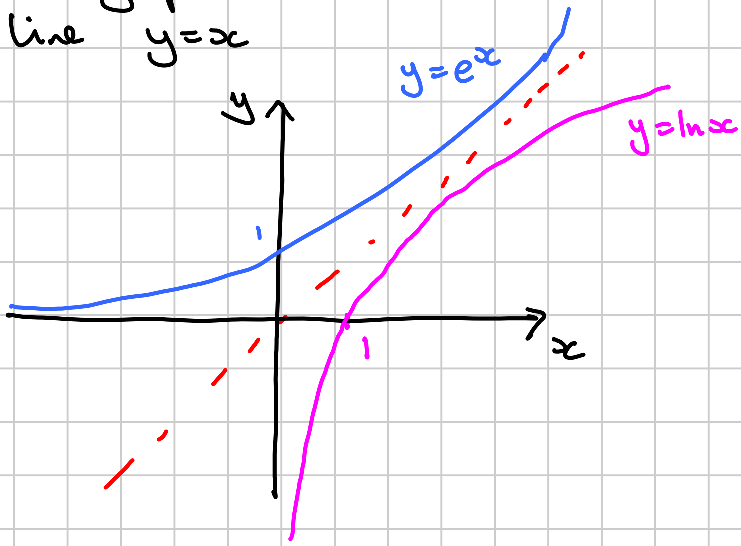
$$e^{\ln\left(\frac{x+1}{2}\right)} = e^3$$

$$\frac{x+1}{2} = e^3$$

$$x = 2e^3 - 1$$

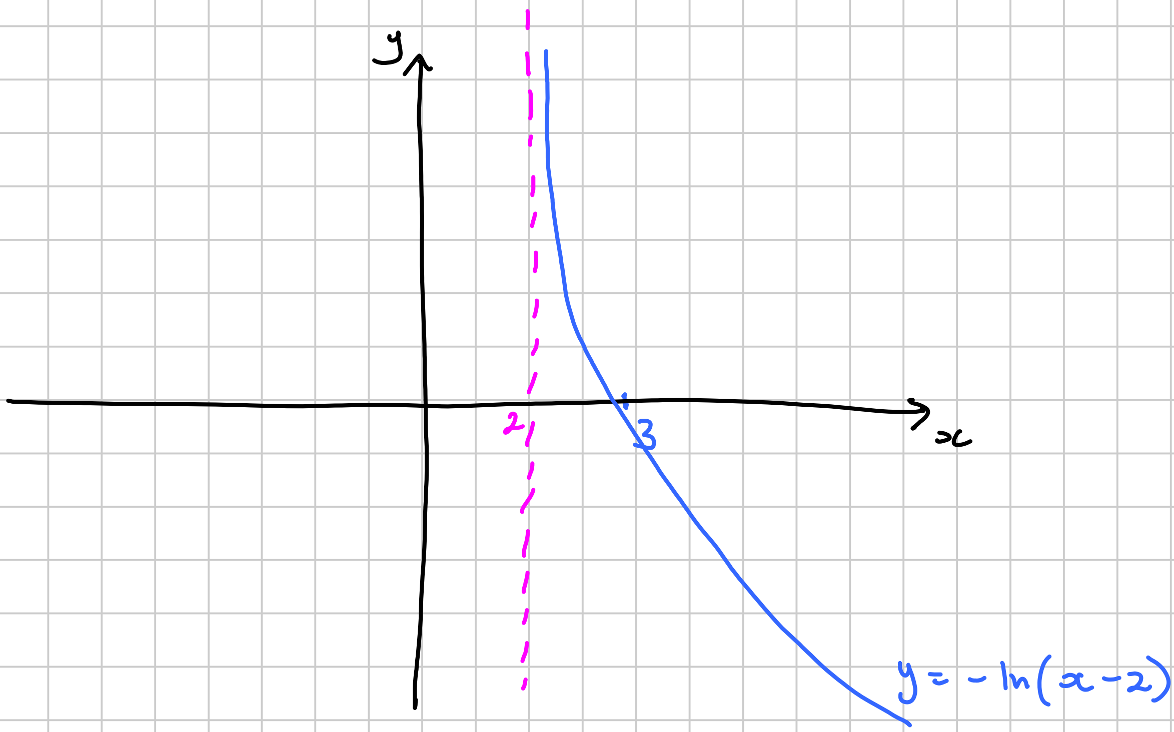
Example 4 Sketch the graph of $y = -\ln(x-2)$.

The graph of $\ln x$ is the reflection of $y = e^x$ in the line $y = x$




$y = \ln(x-2)$ is translated 2 to the right

$y = -\ln(x-2)$ is reflected in the x-axis




p 36 Ex 3B Q 1, 2, 3acf, 4, 5, 6, 7