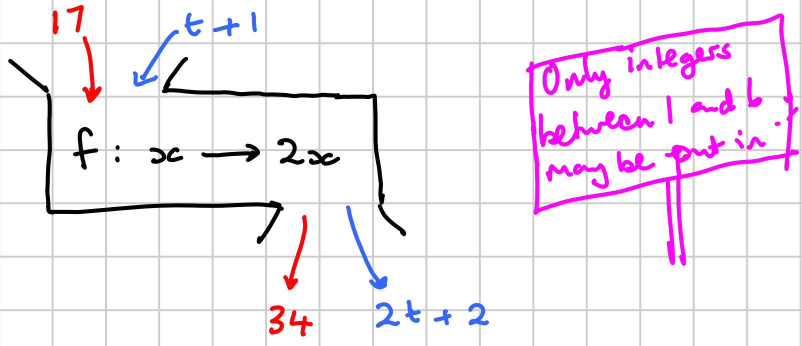
Functions

A function is a rule for changing one number into another. We can think of it as a 'function machine':

e.g.



We write this as $f: x \to 2x$ or $f(x) = 2x$

To show the result of putting in a particular input we write e.g. $f(17) = 34$

$f(t+1) = 2t + 2$

Domain and Range

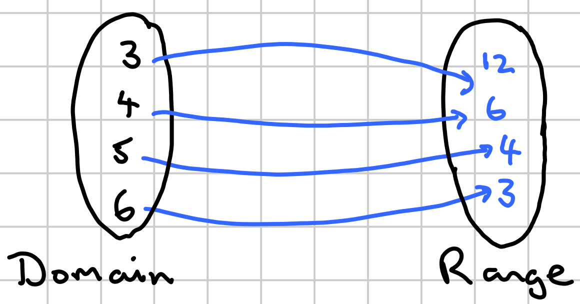
The DOMAIN of a function is the set of numbers which we are allowed to put into it.

The RANGE of a function is the set of values which can be output using the given domain

Mapping Diagrams

If the domain of the function consists of just a few numbers, we can illustrate the function using a mapping diagram.

e.g. Draw a mapping diagram for the function $h: x \to \frac{12}{x-2}$ with the domain $\{3, 4, 5, 6\}$

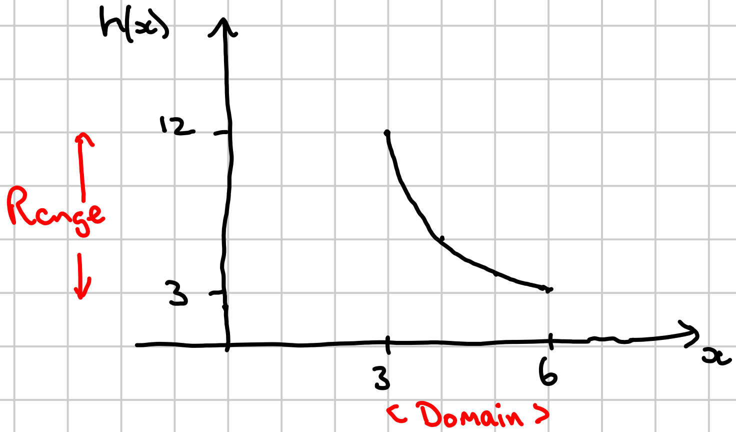


Graphs

If the domain of the function is a continuous interval of numbers, we can illustrate it using a graph

e.g. Draw a graph of the function $h : x \rightarrow \frac{12}{x-2}$

with domain $3 \leq x \leq 6$



The range of the function is

$3 \leq h(x) \leq 12$

Note:
- The domain is the interval 'covered' by the graph on the x-axis
- The range is the interval 'covered' by the graph on the y-axis.

Many-to-one or One-to-one

If different inputs to a function can give the same output, the function is called MANY-TO-ONE.

e.g. If $g : x \rightarrow x^2 + 5$ with domain $x \in \mathbb{R}$

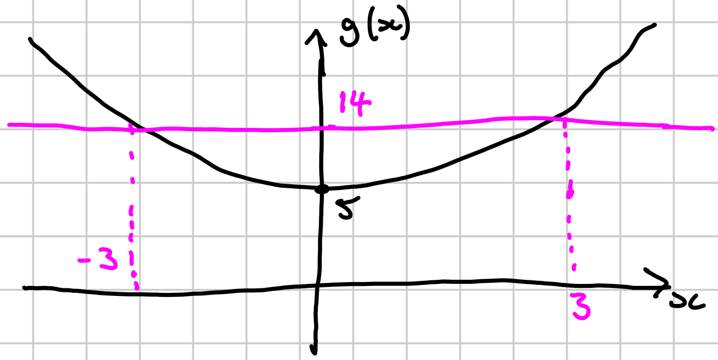
g is MANY-TO-ONE because $g(3) = 14$

$g(-3) = 14$

If each input give a unique output, the function is ONE-TO-ONE.

e.g. h in the previous example is one-to-one.

Graph of $g(x)$:



The range is

$g(x) \geq 5$

A horizontal line drawn across the graph of $g(x)$ crosses in two places.
A horizontal line drawn across the graph of $h(x)$ (above) crosses it in just one place, so h is a ONE-TO-ONE function.

[If a vertical line crosses in more than one place the graph is ONE-TO-MANY but functions are not allowed to be one-to-many.

so $x \to \pm\sqrt{x}$ is <u>not</u> a function
$x \to +\sqrt{x}$ is a function]

Restricting the domain of a function.

- Sometimes the domain of a function is $x \in \mathbb{R}$ ie, x can be any real number.
- Sometimes we need to restrict the domain, because certain values do not 'work'. eg for the function h above, the domain cannot include the value 2, because $\frac{12}{2-2}$ is impossible.
- Sometimes we need to restrict the domain if we want a function to be one-to-one.
  e.g g as defined above is many-to-one.
  If we change the domain of g to $\{x \geq 0\}$, g is now one-to-one.

Composition of functions

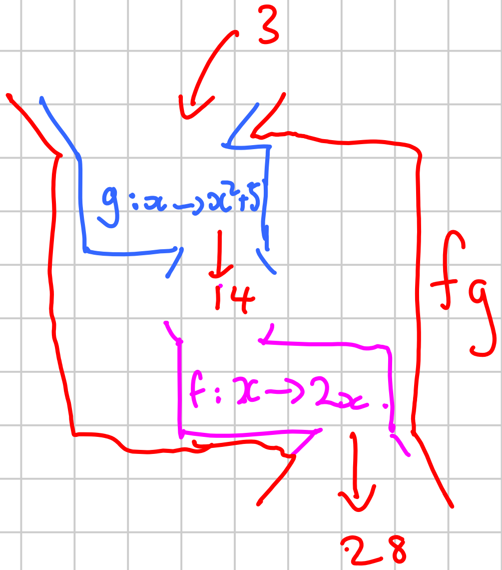
The function $fg(x)$ is defined as $f(g(x))$.

ie, "starting with x, do g to it, then do f to the result"

So if $f : x \to 2x$
$g : x \to x^2 + 5$

$fg(3) = f(14) = 28$
$gf(3) = g(6) = 41$

$f^2(3) = ff(3) = f(6) = 12$
$g^2(3) = gg(3) = g(14) = 201$



Writing fg as a single function

We need to work out $f(g(x))$:

$$f(g(x)) = f(x^2+5)$$
$$= 2(x^2+5) \quad \text{or} \quad 2x^2+10$$

ie/ $fg : x \rightarrow 2x^2+10$

(Check: $fg(3) = 2\times 9+10 = 28$ as above)

$$g(f(x)) = g(2x)$$
$$= (2x)^2+5$$
$$= 4x^2+5$$

So $gf : x \rightarrow 4x^2+5$

(Check: $gf(3) = 4\times 9+5 = 41$ as above)

Ex 1.3
Q 1 abdefh, 3a, 7, 8

Ex 1.4 Q 1

If not here Monday

Inverses

The inverse of a function f is written f^{-1} (don't mix this up with f' which is the derivative of f).
f^{-1} is the function which 'undoes' the effect of f.

To find f^{-1}:
- write $y = f(x)$
- change the subject to x
- swap the 'x's with the 'y's and vice versa.

e.g. to find g^{-1}

$$y = x^2+5$$
$$y-5 = x^2$$
$$\sqrt{y-5} = x$$

So $g^{-1} : x \rightarrow +\sqrt{x-5}$ (domain $x \geq 5$)

<u>Note</u>
- The <u>range</u> of g is the <u>domain</u> of g^{-1}. (this is the case for any function)
- The domain of g is the range of g^{-1}.

BUT g is many-to-one e.g $g(3)=14$, $g(-3)=14$

So g^{-1} would be one-to-many but this is not allowed eg. is $g^{-1}(14) = 3$ or $g^{-1}(14)=-3$?

So we have to restrict the domain of g to make it one-to-one.

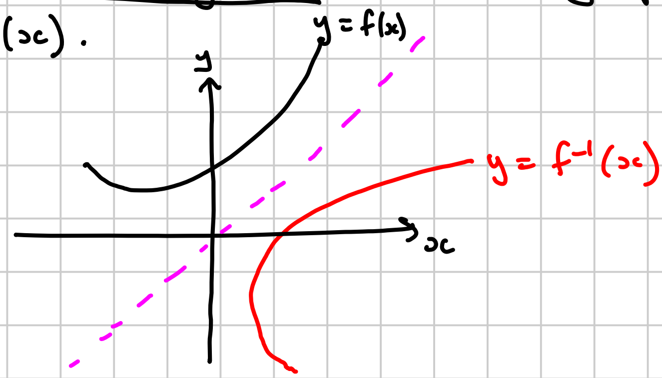
$g: x \rightarrow x^2+5$ with domain $x \geqslant 0$
and range $g(x) \geqslant 5$

g^{-1} $x \rightarrow +\sqrt{x-5}$ with domain $x \geqslant 5$
and range $g^{-1}(x) \geqslant 0$

To have an inverse, a function must be one-to-one. If it isn't we need to restrict its domain to make it one-to-one.

The graph of an inverse function

Because for an inverse function the rôles of x and y are reversed, the graph of $y=f^{-1}(x)$ is the REFLECTION in the line $y=x$ of the graph of $y=f(x)$.



More examples

① Find the inverse of

$$f : x \to \frac{x-3}{x+4} \qquad (x \in \mathbb{R},\ x \neq -4)$$

$$y = \frac{x-3}{x+4}$$

$$y(x+4) = x-3$$
$$yx + 4y = x - 3$$
$$yx - x = -4y - 3$$
$$x(y-1) = -4y-3$$
$$x = \frac{-4y-3}{y-1} = \frac{4y+3}{1-y}$$

$$f^{-1} : x \to \frac{4x+3}{1-x} \qquad (x \in \mathbb{R},\ x \neq 1)$$

② If $f : x \to \dfrac{x-3}{x+4}$ and $g : x \to \dfrac{1}{x+2}$
$(x \neq -2)$

Find fg.

$$fg(x) = f(g(x))$$
$$= f\left(\frac{1}{x+2}\right)$$
$$= \frac{\frac{1}{x+2} - 3}{\frac{1}{x+2} + 4}$$
$$= \frac{1 - 3(x+2)}{1 + 4(x+2)}$$
$$= \frac{-3x-5}{4x+9}$$

$$fg : x \to \frac{-3x-5}{4x+9} \qquad \left(x \neq -\tfrac{9}{4}\right)$$

p 14 Ex 2B Q 3, 4
2C Q 2, 3, 4, 7
2D Q 2, 3, 4, 6, 7, 8
2E Q (all)