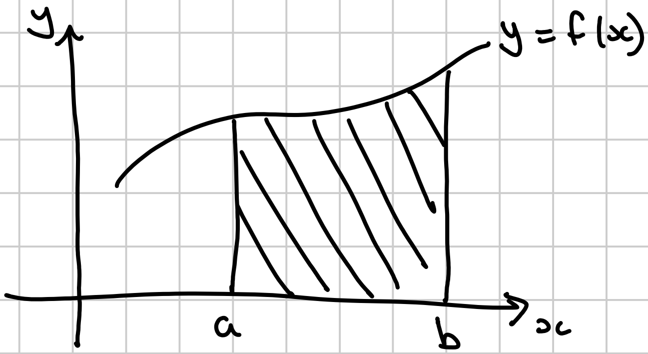
Integration – finding the area under a curve

Note Title 22/03/2012

The integral of a function $f(x)$ is written $F(x)$.
or $\int y \, dx$.

The area under a curve $y = f(x)$, between two vertical lines $x = a$ and $x = b$, is given by



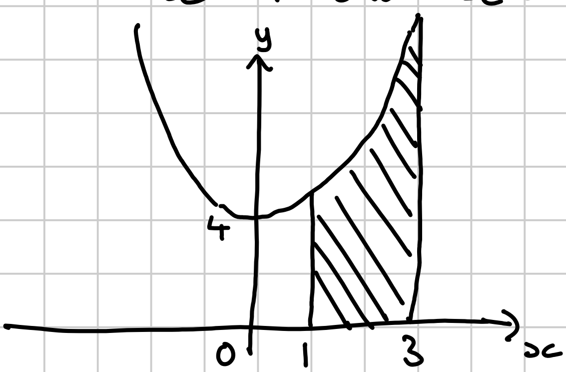
$$\boxed{F(b) - F(a)}$$

This can also be written as

$$\boxed{\int_a^b f(x) \, dx}$$

Examples

(1) Find the area under the curve $y = x^2 + 4$ between $x = 1$ and $x = 3$.



$f(x) = x^2 + 4$

$F(x) = \frac{1}{3}x^3 + 4x + c$

$F(3) - F(1) =$

$(9 + 12 + c) - (\frac{1}{3} + 4 + c)$

$= 16\frac{2}{3}$ sq units

Since the "+c"s always cancel out, we don't need to write them when doing integration to find an area.

Alternative notation:

$$\int_1^3 x^2 + 4 \, dx = \left[\frac{1}{3}x^3 + 4x\right]_1^3$$

numbers to substitute then subtract

$$= (9 + 12) - (\frac{1}{3} + 4)$$

$$= \underline{\underline{16^2/_3}}$$

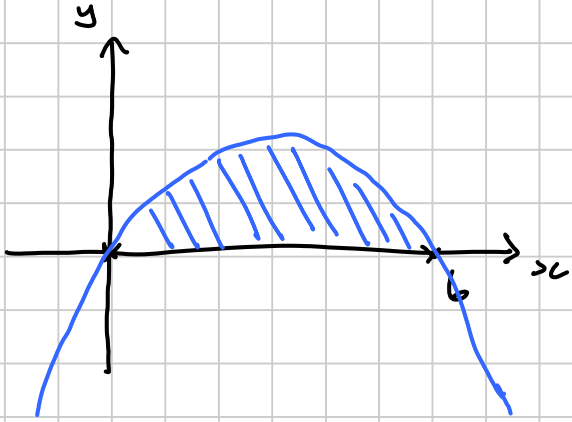
② Find the area enclosed between the x-axis and the curve $y = 6x - x^2$.

Sketch the curve; if $x = 0$, $y = 0$

if $y = 0$, $6x - x^2 = 0$

$x(6 - x) = 0$

$x = 0$ or $x = 6$



$$\text{Area} = \int_0^6 6x - x^2 \, dx$$

$$= \left[3x^2 - \tfrac{1}{3}x^3\right]_0^6$$

$$= (108 - 72) - (0 - 0)$$

$$= \underline{\underline{36 \text{ sq units}}}$$

③ Find the area enclosed between the curve $y = 3x^2 - 3$ and the x-axis.

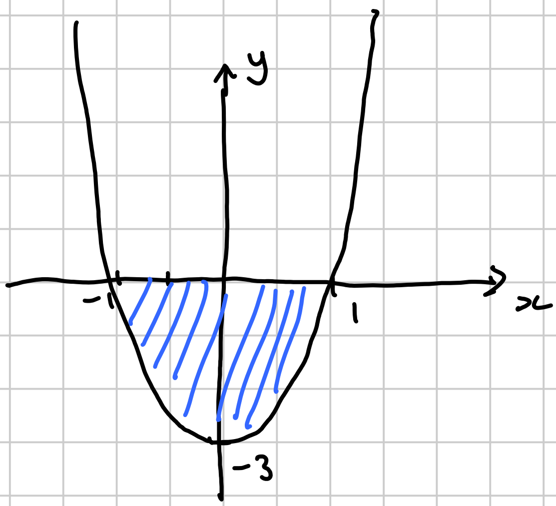
Sketch the curve: if $x = 0$ $y = -3$

if $y = 0$, $3x^2 - 3 = 0$

$3x^2 = 3$

$x^2 = 1$

$x = 1$ or -1



$$\text{Area} = \int_{-1}^{1} 3x^2 - 3 \, dx$$

$$= \left[x^3 - 3x\right]_{-1}^{1}$$

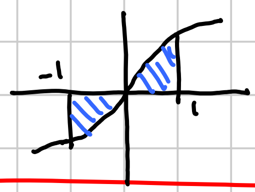
$$= (1 - 3) - (-1 + 3)$$

$$= -4$$

ie, 4 sq units but below the x-axis

Note: Areas below the x-axis come out as negative.

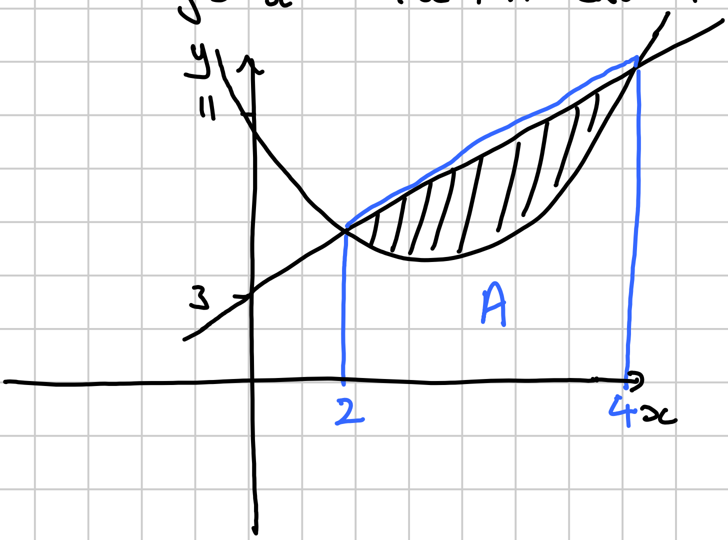
So if we are asked to find an area as shown:



then we need to find the area below the axis and the area above the axis separately p161 Ex 11B Q 1 abce, 2, 3, 4, 5

need to sketch the graph for these.

(4) Find the area enclosed between the curve $y = x^2 - 4x + 11$ and the line $y = 2x + 3$.

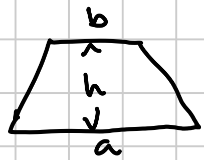


To find where the line crosses the curve:

$$x^2 - 4x + 11 = 2x + 3$$
$$x^2 - 6x + 8 = 0$$
$$(x-4)(x-2) = 0$$
$$x = 4 \quad \text{or} \quad x = 2$$
$$y = 11 \qquad\qquad y = 7$$

Shaded area = area of trapezium − area A under curve

<u>**Method 1**</u> Area of trapezium $= \frac{1}{2}(a+b)h$



$$= \frac{1}{2}(11+7)2$$
$$= 18$$

$$\text{Area } A = \int_2^4 x^2 - 4x + 11 \, dx$$
$$= \left[\frac{1}{3}x^3 - 2x^2 + 11x\right]_2^4$$
$$= \left(\frac{64}{3} - 32 + 44\right) - \left(\frac{8}{3} - 8 + 22\right)$$
$$= 16\tfrac{2}{3}$$

Shaded area $= 18 - 16\frac{2}{3} = \underline{\underline{1\frac{1}{3} \text{ sq units}}}$

Method 2 Area of trapezium $= \int_2^4 2x + 3 \, dx$

$$= \left[x^2 + 3x\right]_2^4$$

$$= (16 + 12) - (4 + 6)$$

$$= 18$$

Area A $= 16\frac{2}{3}$ (as above)

Shaded area $= 18 - 16\frac{2}{3} = \underline{\underline{1\frac{1}{3} \text{ sq units}}}$

Method 3

Shaded area $= \int_2^4 2x+3 \, dx - \int_2^4 x^2 - 4x + 11 \, dx$

$$= \int_2^4 2x + 3 - (x^2 - 4x + 11) \, dx$$

$$= \int_2^4 6x - x^2 - 8 \, dx$$

$$= \left[3x^2 - \frac{1}{3}x^3 - 8x\right]_2^4$$

$$= \left(48 - \frac{64}{3} - 32\right) - \left(12 - \frac{8}{3} - 16\right)$$

$$= \underline{\underline{1\frac{1}{3}}}$$

Note: In method 3 we have used one of the rules about integrals:

(Provided the limits are the same:)

$$\boxed{\int f(x) \, dx \pm \int g(x) \, dx = \int f(x) \pm g(x) \, dx}$$

Another rule is:

$$\int k\, f(x)\,dx = k \int f(x)\,dx$$ where k is a constant

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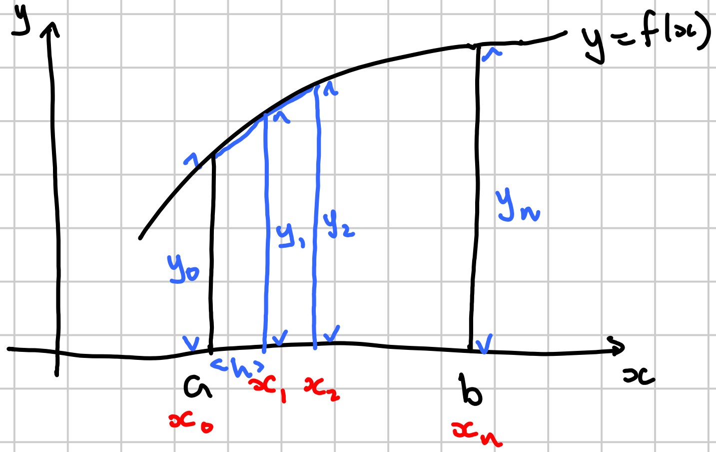
The Trapezium Rule

If we do not know how to integrate a function algebraically, we can find an approximation to the area numerically.



We divide the area into n strips each of width $\boxed{h = \frac{b-a}{n}}$ using $n+1$ vertical lines.

The x-coordinates of the lines are
$x_0 = a$
$x_1 = a+h$
.....
$x_n = b$

The heights of the lines are
$y_0 = f(x_0)$
$y_1 = f(x_1)$
....
$y_n = f(x_n)$.

We approximate each strip by a trapezium.

Area of first trapezium $= \frac{1}{2}h(y_0 + y_1)$
" " second " $= \frac{1}{2}h(y_1 + y_2)$
" " third " $= \frac{1}{2}h(y_2 + y_3)$
etc... " " last " $= \frac{1}{2}h(y_{n-1} + y_n)$

So total area under curve $\approx \frac{1}{2}h\left(y_0 + y_1 + y_1 + y_2 + y_2 + y_3 + \cdots y_{n-1} + y_n\right)$

$$\boxed{\int_a^b f(x)\,dx \approx \frac{1}{2}h\left(y_0 + 2(y_1 + y_2 + \cdots y_{n-1}) + y_n\right)}$$

This is the trapezium rule. It is in the formula book.

Example Find the approximate value of $\int_1^4 2^x \, dx$ using the trapezium rule with 6 strips

$$h = \frac{4-1}{6} = 0.5$$

Table:

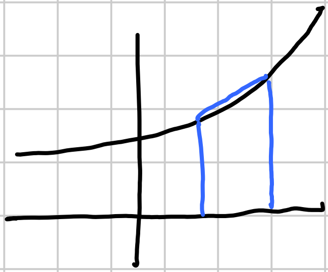
	x_0	x_1					x_n
x	1	1.5	2	2.5	3	3.5	4
y	2	2.8284	4	5.6568	8	11.3137	16
	y_0	y_1					y_n

(6 strips so 7 numbers in table)

$$\int_1^4 2^x \, dx \approx \frac{1}{2} \times 0.5 \left(2 + 2(2.8284 + \cdots + 11.3137) + 16\right)$$

$$\approx \underline{\underline{20.4}} \quad (1 \text{ dp})$$

(b) Is this an overestimate or an underestimate?



Area of each trapezium is greater than area under curve (curve is CONCAVE)

$\Rightarrow$ OVERESTIMATE.

(c) How could the estimate be improved?

By using more strips!

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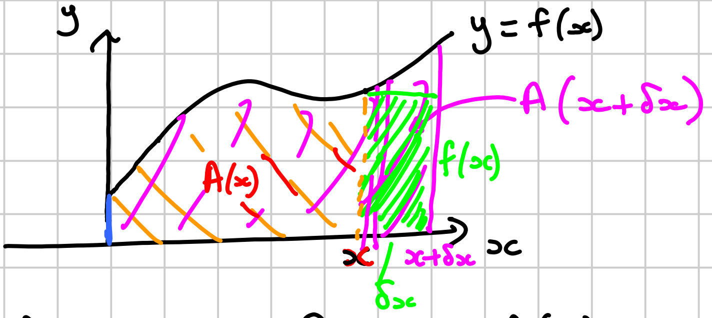
To show that the area under a curve is given by the reverse process to differentiation :-

Suppose we wish to find the area between a curve and the x-axis.



We define a new function $A(x)$ as the area under the curve between a left-hand boundary which we fix at a certain point (which we choose), and a variable right hand boundary x.
(In this diagram the left hand boundary has been fixed at $x = 0$)

We now try to find $A'(x)$
Using the general formula for a derivative (see C1) :

$$A'(x) = \lim_{\delta x \to 0} \left(\frac{A(x+\delta x) - A(x)}{\delta x} \right)$$

The numerator of this fraction is the difference between the area shaded purple and the area shaded red on the diagram, which is a thin strip (becoming thinner as $\delta x \to 0$)
We can approximate this strip as a rectangle of width δx and height $f(x)$.
As $\delta x \to 0$, this approximation becomes exact.

So $$A'(x) = \lim_{\delta x \to 0} \left(\frac{f(x)\cancel{\delta x}}{\cancel{\delta x}} \right)$$

$$= \lim_{\delta x \to 0} (f(x))$$

$$= f(x)$$

So $A(x)$ must be the same function as $F(x)$. This is the "Fundamental Theorem of Calculus" - that the process of finding an area is the inverse of the process of finding a gradient.

Now $F(x)$ contains an arbitrary constant. Different values of this constant correspond to different choices of the left hand boundary of the area.
But in practice we do not need to think about where the left hand boundary is since we find an area under $f(x)$ between $x = a$ and $x = b$ by calculating

$$\boxed{F(b) - F(a)}$$