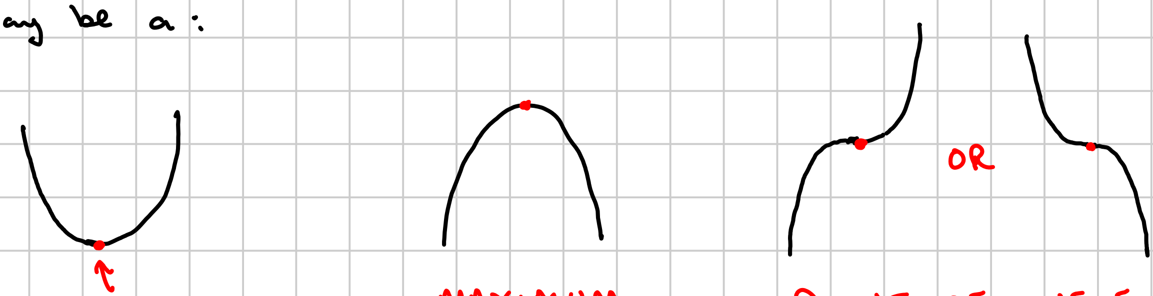
Note Title 15/03/2012

Differentiation

Turning Points (Stationary Points)

A TP is a point where the gradient of a curve is zero.
It may be a:



MINIMUM MAXIMUM POINT OF INFLEXION

In order to tell what type of turning point we have, we can work out $\frac{d^2y}{dx^2}$ (ie, $f''(x)$) and substitute in the value of x at the TP.

$$\boxed{\begin{array}{lll} \frac{d^2y}{dx^2} > 0 & \Rightarrow & \text{TP is a MINIMUM} \\ \frac{d^2y}{dx^2} < 0 & \Rightarrow & \text{TP is a MAXIMUM} \\ \frac{d^2y}{dx^2} = 0 & \Rightarrow & \text{We CAN'T TELL the type of TP} \end{array}}$$

Examples

① Find the turning points of $y = x^3 - 3x$ and determine their nature. Hence sketch the curve.

$$\frac{dy}{dx} = 3x^2 - 3$$

We want $3x^2 - 3 = 0$

$(\div 3)$ $x^2 - 1 = 0$

$(x+1)(x-1) = 0$

$x = -1$ or $x = 1$

$y = (-1)^3 - 3(-1) = 2$ $\quad$ $y = 1^3 - 3(1) = -2$

$$\frac{d^2y}{dx^2} = 6x.$$

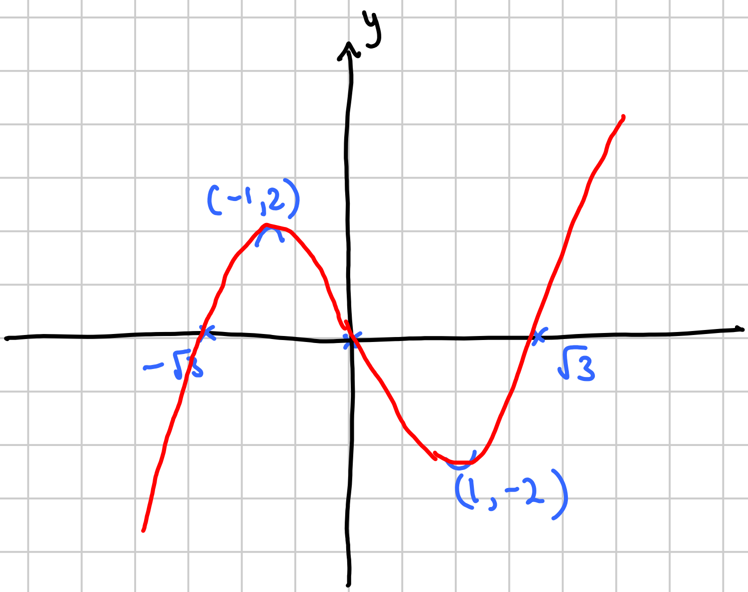
If $x = -1$, $\frac{d^2y}{dx^2} = -6 < 0 \Rightarrow$ MAX at $(-1, 2)$

If $x = 1$ $\frac{d^2y}{dx^2} = 6 > 0 \Rightarrow$ MIN at $(1, -2)$

To sketch the graph find where it crosses the axes.

If $x = 0$, $y = 0$

If $y = 0$,
$$x^3 - 3x = 0$$
$$x(x^2 - 3) = 0$$
$$x(x + \sqrt{3})(x - \sqrt{3}) = 0$$
$$x = 0 \quad \text{or} \quad x = -\sqrt{3} \quad \text{or} \quad x = \sqrt{3}$$



② Find the turning points of $y = x^5 - 10x^4 + 25x^3$.

$$\frac{dy}{dx} = 5x^4 - 40x^3 + 75x^2$$

We want
$$5x^4 - 40x^3 + 75x^2 = 0$$
$$5x^2(x^2 - 8x + 15) = 0$$
$$5x^2(x - 3)(x - 5) = 0$$

$x = 0$, $y = 0$ or $x = 3$, $y = 108$ or $x = 5$, $y = 0$

$$\frac{d^2y}{dx^2} = 20x^3 - 120x^2 + 150x$$
$$= 10x(2x^2 - 12x + 15)$$

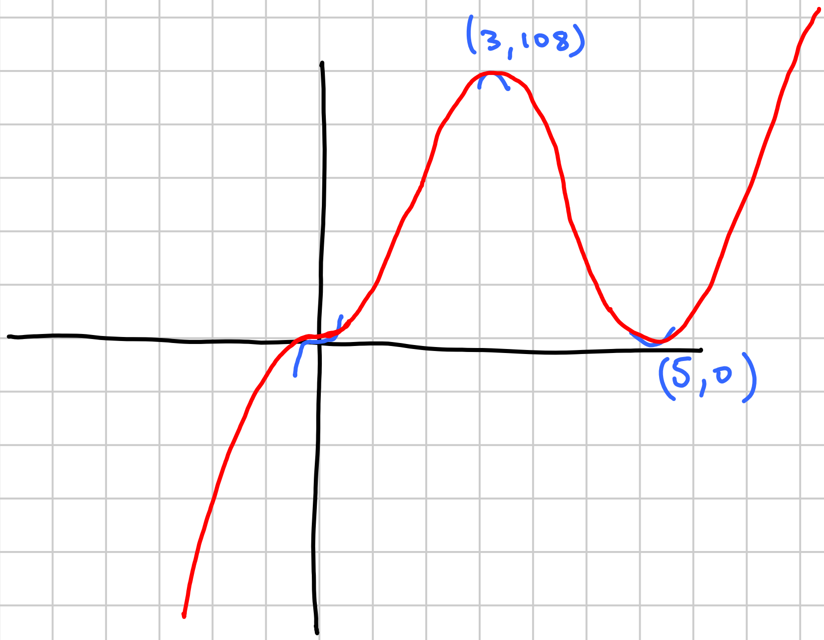
If $x = 0$, $\frac{d^2y}{dx^2} = 0 \Rightarrow$ CAN'T TELL

$x = -1$ gradient $= 120$ | $x = 0$ $g = 0$ | $x = 1$ $g = 40$

/ — / $\Rightarrow$ POINT OF INFLEXION. at $(0, 0)$

If $x = 3$, $\frac{d^2y}{dx^2} = -90 < 0 \Rightarrow$ MAX at $(3, 108)$

If $x = 5$ $\frac{d^2y}{dx^2} = 5 > 0 \Rightarrow$ MIN at $(5, 0)$



p134 Ex 9B Q 3 bcefg Finish for HWK.

(3) Find the turning point of $y = \sqrt{x} + \frac{4}{x}$ and determine its nature. Hence sketch the curve.

$$y = x^{1/2} + 4x^{-1}$$

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2} - 4x^{-2}$$

We want $\frac{1}{2}x^{-1/2} - 4x^{-2} = 0$

$$x^{-1/2} - 8x^{-2} = 0$$

$$\frac{1}{\sqrt{x}} - \frac{8}{x^2} = 0$$

$$\frac{1}{\sqrt{x}} = \frac{8}{x^2}$$

$$\frac{x^2}{\sqrt{x}} = 8$$

$$x^{3/2} = 8 \quad \rightarrow \quad (x^{3/2})^{2/3} = 8^{2/3}$$

$$x = 8^{2/3}$$

$$(x^{1/2})^3 = 8$$

$$x^{1/2} = 2$$

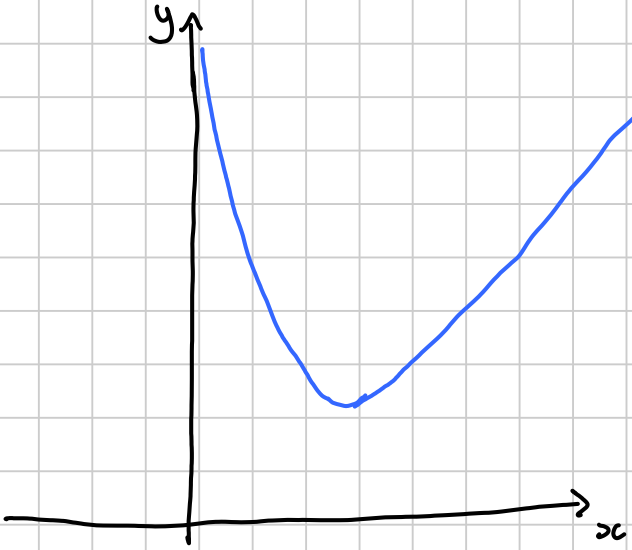
$$x = 4$$

$$\frac{d^2y}{dx^2} = -\frac{1}{4}x^{-3/2} + 8x^{-3}$$

If $x = 4$, $\frac{d^2y}{dx^2} = -\frac{1}{4} \times \frac{1}{8} + 8 \times \frac{1}{64}$

$$= -\frac{1}{32} + \frac{1}{8}$$

$$= \frac{3}{32} > 0 \Rightarrow \text{MIN at } (4, 3)$$



Curve doesn't exist if $x < 0$.

$x = 0$ is an asymptote (since $\frac{4}{x} \to \infty$ as $x \to 0$)

3. A cylindrical can has a volume of 1000 cm^3.

(a) Show that the surface area is given by the formula $A = 2\pi r^2 + \frac{2000}{r}$

(b) Find the minimum surface area of the can.

(a) We know that A = area of top and base + curved area

$$= 2\pi r^2 + 2\pi r h$$

But this involves 2 variables, r and h.

They are not independent, because the volume is fixed – so if we increase r we must reduce h.

We can express h in terms of r:

$$V = \pi r^2 h$$

$$\Rightarrow \quad h = \frac{V}{\pi r^2}$$

$$\Rightarrow \quad h = \frac{1000}{\pi r^2} \qquad \text{since } V = 1000.$$

We substitute this into the formula for A:

$$A = 2\pi r^2 + 2\cancel{\pi}\cancel{r} \times \frac{1000}{\cancel{\pi}\cancel{r^2}\, r}$$

$$A = 2\pi r^2 + \frac{2000}{r}$$

(b) $$\frac{dA}{dr} = 4\pi r - 2000r^{-2}$$

We want
$$4\pi r - \frac{2000}{r^2} = 0$$
$$4\pi r^3 - 2000 = 0$$
$$4\pi r^3 = 2000$$
$$r^3 = \frac{2000}{4\pi} = \frac{500}{\pi}$$
$$r = \sqrt[3]{\frac{500}{\pi}}$$
$$= \underline{\underline{5.42\text{cm}}}$$

[To check this is a minimum:

$$\frac{d^2A}{dr^2} = 4\pi + \frac{4000}{r^3}$$

If $r = 5.42$, $\frac{d^2A}{dr^2} > 0 \Rightarrow$ MIN.]

Min surface area is $A = 2\pi(5.42)^2 + \frac{2000}{5.42}$

$= \underline{\underline{554\text{ cm}^2}}$

[Note that $h = \frac{1000}{\pi(5.42)^2} = 10.84\text{ cm}$.

ie, height = diameter]

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Increasing and Decreasing Functions

A function is INCREASING in a certain interval if $\frac{dy}{dx} > 0$ in that interval. It is DECREASING if $\frac{dy}{dx} < 0$.

e.g. in example 1 above, if $f(x) = x^3 - 3x$, then $f(x)$ is decreasing in the interval $-1 < x < 1$.
and increasing in the intervals $\{x < -1\} \cup \{x > 1\}$

4 If $f(x) = x^3 + 3x - 7$, show that $f(x)$ is increasing for all $x \in \mathbb{R}$. (ie for all values of x)

$$f'(x) = 3x^2 + 3$$

Since $x^2 \geq 0$ for all values of x, $f'(x) \geq 3$ for all values of x. So $f(x)$ is increasing for all values of x

Ex 9A 1adf 2abd