BINOMIAL THEOREM

Combinations

The number of ways of choosing r objects from n is written $_nC_r$ or ^nC_r or $\binom{n}{r}$.

This can be worked out is various ways:—

① $$^nC_r = \frac{\overbrace{n(n-1)(n-2)\ldots\ldots n-(r-1)}^{r\text{ terms}}}{r!}$$

e.g $$^8C_3 = \frac{8 \times 7 \times \cancel{6}}{\cancel{3} \times \cancel{2} \times 1} = 56$$

② Pascal's Triangle is made of ^nC_r s

$$\begin{array}{ccccccccc}
 & & & & {}^0C_0 & & & & \\
 & & & {}^1C_0 & & {}^1C_1 & & & \\
 & & {}^2C_0 & & {}^2C_1 & & {}^2C_2 & & \\
 & {}^3C_0 & & {}^3C_1 & & {}^3C_2 & & {}^3C_3 & \\
{}^4C_0 & & {}^4C_1 & & {}^4C_2 & & {}^4C_3 & & {}^4C_4
\end{array}
\qquad
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}$$

③ Use ^nC_r on calculator. e.g. $^{20}C_4 = 4845$

NB $^nC_{n-r}$ is always equal to ^nC_r

e.g. $^{20}C_{16} = {}^{20}C_4$ etc.

Binomial Theorem

$$(a+b)^n = a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + \ldots\ldots\ldots b^n$$

Examples

① $(2x + 3y)^4$ (use Pascal's triangle)

$$(2x)^4 + {}^4C_1(2x)^3(3y) + {}^4C_2(2x)^2(3y)^2 + {}^4C_3(2x)(3y)^3 + (3y)^4$$

$$= 16x^4 + 4 \times 8x^3 \times 3y + 6(4x^2)(9y^2) + 4(2x)(27y^3) + 81y^4$$

$$= 16x^4 + 96x^3y + 216x^2y^2 + 216xy^3 + 81y^4$$

(2) Find the first 3 terms of $(2-3x)^{10}$

$$2^{10} + {}^{10}C_1\, 2^9(-3x) + {}^{10}C_2\, 2^8(-3x)^2$$

$$1024 + 10 \times 512 \times (-3x) + \frac{10 \times 9}{2 \times 1} 256(9x^2)$$

$$= 1024 - 15360x + 103680x^2$$

(3) Find the term in x^7 in the expansion of $(1+2x)^{20}$

$${}^{20}C_7(1)^{13}(2x)^7$$

$$= 77520 \times 128x^7$$

$$= 9922560x^7$$

p75 Ex 5C Q 1 agh (use Pascals Δ)
2 bfh (use $\frac{n(n-1)\cdots}{r!}$)
3 bfg (use ${}^nC_r$ button)

If x is small (close to 0), high powers of x will be smaller still. So in this case we can get a good approximation using just the first few terms of an expansion.

(4) Use the answer to (2) to find an approximate value for 1.97^{10}.

If $2-3x = 1.97$, $3x = 0.03$
$x = 0.01$

$$(2-3 \times 0.01)^{10} \approx 1024 - 15360 \times 0.01 + 103680 \times 0.0001$$

$$\approx 880.768$$

$1.97^{10} \approx \underline{\underline{880.768}}$ (true answer 880.36 (2dp))

⑤ Expand $(2-x)(1+2x)^3$

$= (2-x)\left(1^3 + {}^3C_1(1)^2(2x) + {}^3C_2(1)(2x)^2 + (2x)^3\right)$

$= (2-x)(1 + 6x + 12x^2 + 8x^3)$

$= 2 + 12x + 24x^2 + 16x^3$
$\quad - x - 6x^2 - 12x^3 - 8x^4$

$= 2 + 11x + 18x^2 + 4x^3 - 8x^4$

⑥ The coefficient of x^2 in $(2+ax)^{10}$ is 1280

Find the possible values of a.

"x^2" term is ${}^{10}C_2(2)^8(ax)^2$

$= 45 \times 256 \times a^2x^2$

Coefficient $= 11520a^2 = 1280$

$a^2 = \frac{1280}{11520} = \frac{1}{9}$

$\underline{\underline{a = \frac{1}{3}}}$ or $\underline{\underline{a = -\frac{1}{3}}}$

⑦ In the expansion of $(4+x)^n$, the coefficient of x^2 is equal to the coefficient of x^3. Find n.

"x^2" term ${}^nC_2 4^{n-2}x^2 = \frac{n(n-1)}{2\times1}4^{n-2}x^2$

"x^3" term ${}^nC_3 4^{n-3}x^3 = \frac{n(n-1)(n-2)}{3\times2\times1}4^{n-3}x^3$

Coefficients are equal: $\frac{\cancel{n(n-1)}}{2}4^{n-2} = \frac{\cancel{n(n-1)}(n-2)}{6}4^{n-3}$

Divide by $n(n-1)$, multiply by 6

$$3 \cdot \frac{\cancel{6}}{\cancel{2}} \times 4^{n-2} = (n-2)\,4^{n-3}$$

(Divide by 4^{n-3})

$$\frac{3 \times 4^{n-2}}{4^{n-3}} = n-2$$

$n-2-(n-3)$
$= n-2-n+3$
$= 1$

$$3 \times 4 = n-2$$

$$\underline{\underline{n = 14}}$$

p 75 Ex 5C Q 4, 5, 7, 8

Ex 5D Q 2, 3, 4