Note Title 06/03/2012

The Binomial Theorem

Permutations and Combinations

The number of ways of arranging n objects in order is $n!$

Example In how many ways can the letters of the word LEWIS be arranged?

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

The number of ways of arranging r objects chosen from n is written ${}^nP_r$

and
$$\boxed{\begin{aligned} {}^nP_r &= \overbrace{n \times (n-1) \times (n-2) \ldots (n-(r-1))}^{r \text{ terms}} \\ &= \frac{n!}{(n-r)!} \end{aligned}}$$

Example The number of ways of choosing a Council Rep, a Charity Rep and a Green Rep from a class of 14 girls is

$$\begin{aligned} {}^{14}P_3 &= 14 \times 13 \times 12 \\ &= \frac{14!}{11!} \\ &= 2184 \end{aligned}$$

The number of ways of choosing r objects from n is written ${}^nC_r$

and
$$\boxed{\begin{aligned} {}^nC_r &= \frac{{}^nP_r}{r!} \\ &= \frac{n \times (n-1) \times (n-2) \times \cdots (n-(r-1))}{r \times (r-1) \times (r-2) \times \cdots 1} \leftarrow r \text{ terms} \\ &= \frac{n!}{(n-r)!\, r!} \end{aligned}}$$

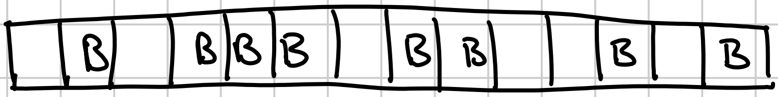
Example A team of 3 is to be chosen from a class of 14 girls. In how many ways can this be done?

$$^{14}C_3 = \frac{14 \times 13 \times 12}{3 \times 2 \times 1} \quad \text{or} \quad \frac{14!}{11!\,3!}$$

$$= \underline{\underline{364}}$$

^nC_r can also be used to find the number of ways of arranging objects of two different types.

Example There are 8 boys and 6 girls in a queue. In how many different orders could they stand?



We can choose the positions in the queue of the 8 boys in $^{14}C_8$ ways. Then the girls fill the remaining spaces.

[Note that we could start by placing the girls, in $^{14}C_6$ ways.]

Solution is $^{14}C_8$ or $^{14}C_6 = \underline{\underline{3003}}$

Note

1. ^nC_r and $^nC_{n-r}$ are always the same.
2. $^nC_0 = 1$ and $^nC_n = 1$
3. $^nC_1 = n$ and $^nC_{n-1} = n$

From the Quincunx sheet, we see that Pascal's Triangle is made up of "^nC_r"s :

$$\begin{array}{ccccccccc} & & & & ^0C_0 & & & & \\ & & & ^1C_0 & & ^1C_1 & & & \\ & & ^2C_0 & & ^2C_1 & & ^2C_2 & & \\ & ^3C_0 & & ^3C_1 & & ^3C_2 & & ^3C_3 & \\ ^4C_0 & & ^4C_1 & & ^4C_2 & & ^4C_3 & & ^4C_4 \end{array} \quad = \quad \begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

etc. etc.

The Binomial Theorem

To work out $(x+y)^5$ we can write:

$$(x+y)(x+y)(x+y)(x+y)(x+y)$$

gives x^5

gives x^4y ie, $xxxxy$. This can be got in ${}^5C_1 = 5$ ways

gives x^3y^2 ie, $xxxyy$ This can be got in ${}^5C_2 = 10$ ways

Similar, x^2y^3 can be got in ${}^5C_3 = 10$ ways

xy^4 can be got in ${}^5C_4$ (or ${}^5C_1$) $= 5$ ways

y^5 can be got in ${}^5C_5 = 1$ way.

So $(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$

Generalising this, we get the Binomial Theorem:

$$\boxed{(x+y)^n = x^n + {}^nC_1 x^{n-1}y + {}^nC_2 x^{n-2}y^2 + \cdots\cdots + y^n}$$

We can get the '${}^nC_r$'s from a calculator or from the n^{th} row of Pascal's Triangle.

Examples

① $(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

② Write down the first 4 terms in the expansion of $(x+y)^{20}$

$$x^{20} + {}^{20}C_1 x^{19}y + {}^{20}C_2 x^{18}y^2 + {}^{20}C_3 x^{17}y^3$$

$$= x^{20} + 20x^{19}y + 190x^{18}y^2 + 1140x^{17}y^3 + \cdots\cdots$$

③ Work out $(2x-y)^4$

$$(2x)^4 + 4(2x)^3(-y) + 6(2x)^2(-y)^2 + 4(2x)(-y)^3 + (-y)^4$$
$$= 16x^4 - 32x^3y + 24x^2y^2 - 8xy^3 + y^4$$

p72 Ex 5A Q 1 bcdefh
p75 Ex 5C Q 2 bdefg

by Monday.

(4) Expand $(2-x)(3+x)^4$

$$(2-x)\left(3^4 + 4(3)^3x + 6(3)^2(x)^2 + 4(3)x^3 + x^4\right)$$
$$= (2-x)(81 + 108x + 54x^2 + 12x^3 + x^4)$$
$$= 162 + 216x + 108x^2 + 24x^3 + 2x^4$$
$$\quad - 81x - 108x^2 - 54x^3 - 12x^4 - x^5$$
$$= \underline{162 + 135x - 30x^3 - 10x^4 - x^5}$$

(5) Find the coefficient of x^3 in the expansion of $(3-2x)(1+x)^{10}$

$$(3-2x)\left(\ldots\ldots\ldots {}^{10}C_2 1^8 x^2 + {}^{10}C_3 1^7 x^3 \ldots\ldots \right)$$
$$= (3-2x)\left(\ldots\ldots\ldots 45x^2 + 120x^3 \ldots\ldots \right)$$
$$=$$

$$\begin{array}{r} 360x^3 \\ -\ 90x^3 \\ \hline 270x^3 \end{array}$$

Coefficient of $x^3 = \underline{\underline{270}}$

(6) Expand $\left(1 + \frac{x}{10}\right)^7$ in ascending powers of x as far as the term in x^3. Hence find an approximation to 1.01^7, to 6dp.

$$1^7 + 7(1)^6\left(\frac{x}{10}\right) + \frac{7\times6}{2\times1}(1)^5\left(\frac{x}{10}\right)^2 + \frac{7\times6\times5}{3\times2\times1}(1)^4\left(\frac{x}{10}\right)^3$$

$$= 1 + \frac{7x}{10} + \frac{21x^2}{100} + \frac{35x^3}{1000} + \ldots$$

If $1 + \frac{x}{10} = 1.01$

$\frac{x}{10} = 0.01$

$x = 0.1$ so subst this into expansion.

$$1 + \frac{0.7}{10} + \frac{0.21}{100} + \frac{0.035}{1000} + \ldots$$

$$= 1 + 0.07 + 0.0021 + 0.000035 + \ldots$$

$$\approx \underline{\underline{1.072135}}$$

(7) In the expansion of $(2+ax)^5$, the coefficient of x^2 is 20. Find the possible values of a.

$${}^5C_2\, 2^3 (ax)^2 = 10 \times 8a^2x^2$$

So $80a^2 = 20$

$a^2 = \frac{20}{80} = \frac{1}{4}$

$\underline{\underline{a = \frac{1}{2} \text{ or } -\frac{1}{2}}}$

(8) In the expansion of $(1+2x)^n$ $(n>0)$, the coefficient of x^2 is 144. Find n.

$${}^nC_2\, 1^{n-2}(2x)^2 = \frac{n(n-1)}{\cancel{2}_1\times 1} \times 1 \times \cancel{4}^2x^2$$

So $2n(n-1) = 144$

$n(n-1) = 72$

$n^2 - n - 72 = 0$

$(n-9)(n+8) = 0$

$\underline{\underline{n = 9}}$ (or $n = -8$ not > 0)

p72 Ex 5A Q 3, 6, 7

p75 Ex 5C Q 4, 6, 7, 8

p76 Ex 5D Q 3, 4, 5.