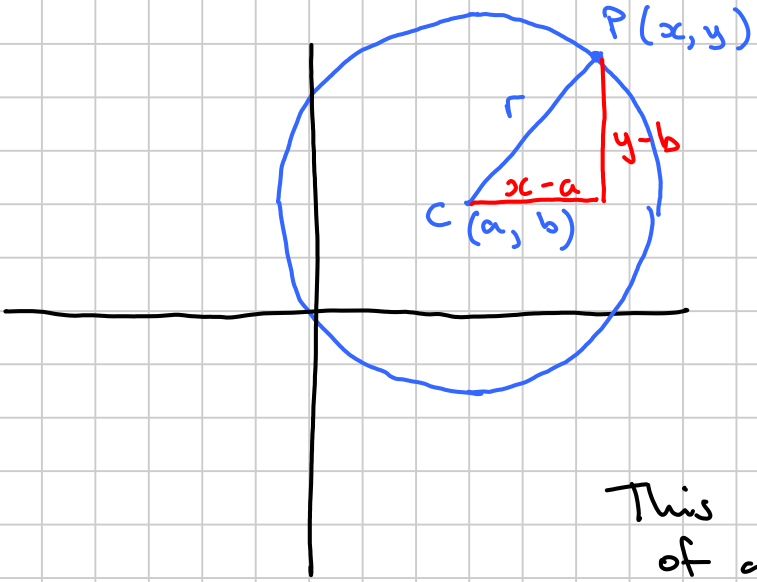
Coordinate Geometry (Circles)



Suppose we have a circle centre C (a, b), radius r.

Then if P (x, y) is <u>any</u> point on the circle,

$$\boxed{(x-a)^2 + (y-b)^2 = r^2}$$

This is the general Cartesian equation of a circle.

Examples

① PQ is the diameter of a circle. P is $(-3, 4)$ Q is $(5, 2)$. Find the equation of the circle.

The centre O of the circle is the midpoint of PQ

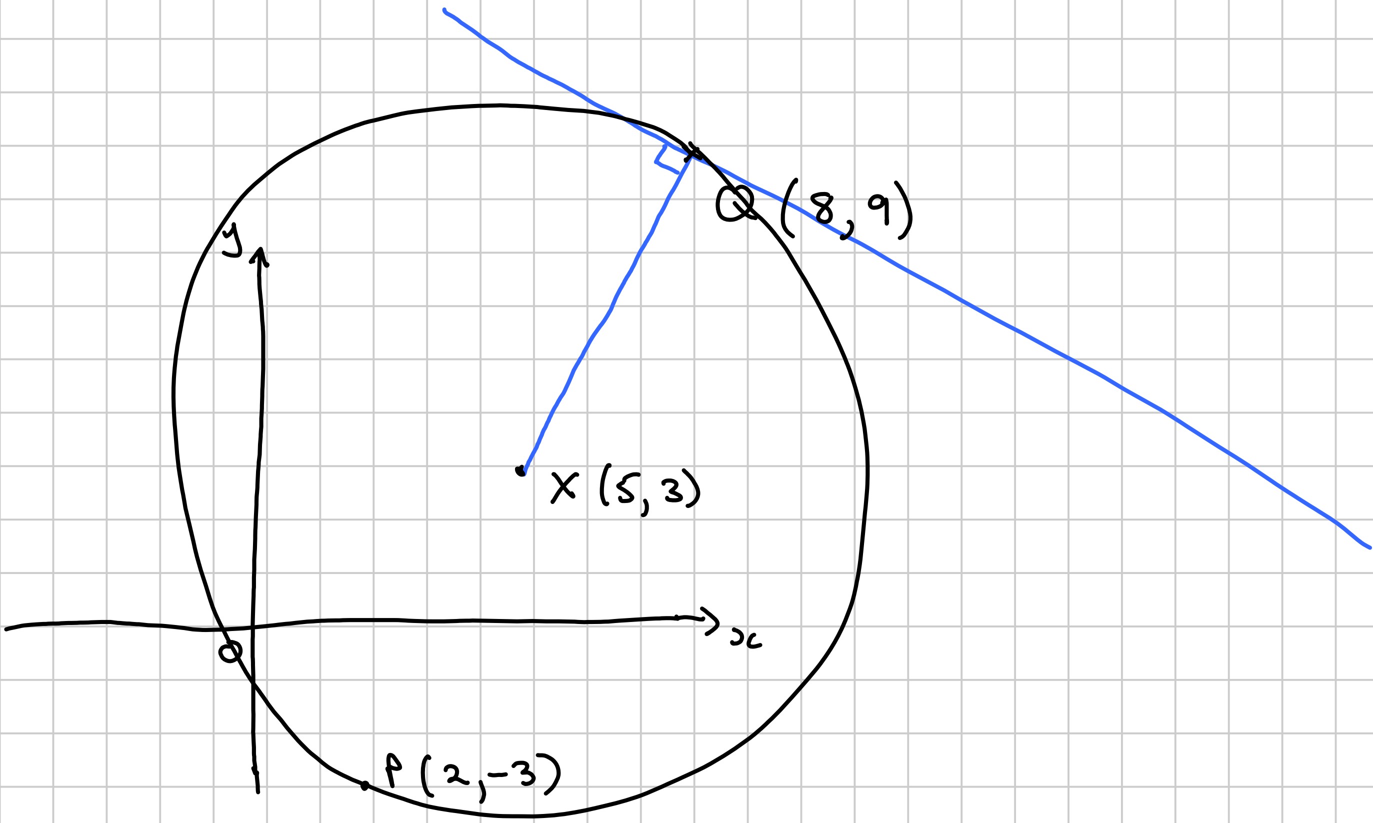
So O is $\left(\frac{-3+5}{2}, \frac{4+2}{2}\right)$ ie/ $(1, 3)$

The radius is the distance OQ

$$OQ = \sqrt{(5-1)^2 + (2-3)^2}$$
$$= \sqrt{4^2 + (-1)^2}$$
$$= \sqrt{17}$$

The equation of the circle is $(x-1)^2 + (y-3)^2 = 17$

② PQ is a diameter of a circle.
P is $(2, -3)$ and the centre of the circle is $X(5,3)$
Find the equation of the tangent to the circle at Q.



To find Q : EITHER $\overrightarrow{PX} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$ so $\overrightarrow{XQ} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$

so Q is $(5+3,\ 3+6)$
$= (8, 9)$

OR Let Q be (c, d)
X is the midpoint of PQ

$\frac{2+c}{2} = 5$ $\qquad$ $\frac{-3+d}{2} = 3$

$c = 8$ $\qquad$ $d = 9$

The tangent is PERPENDICULAR to the radius.

Gradient of radius XQ is $\frac{9-3}{8-5} = \frac{6}{3} = 2$

Gradient of tangent is $-\frac{1}{2}$

Equation of tangent is $y - 9 = -\frac{1}{2}(x - 8)$

$$2y - 18 = -(x - 8)$$

$$\underline{\underline{x + 2y - 26 = 0}}$$

(3) Find the centre and radius of the circle with equation $x^2 + y^2 - 6x + 8y = 0$

COMPLETE THE SQUARE for 'x's and for 'y's

$$\underbrace{x^2 - 6x + 9} - 9 + \underbrace{y^2 + 8y + 16} - 16 = 0$$

$$(x-3)^2 - 9 + (y+4)^2 - 16 = 0$$

$$(x-3)^2 + (y+4)^2 = 25$$

$$(x-3)^2 + (y-(-4))^2 = 25$$

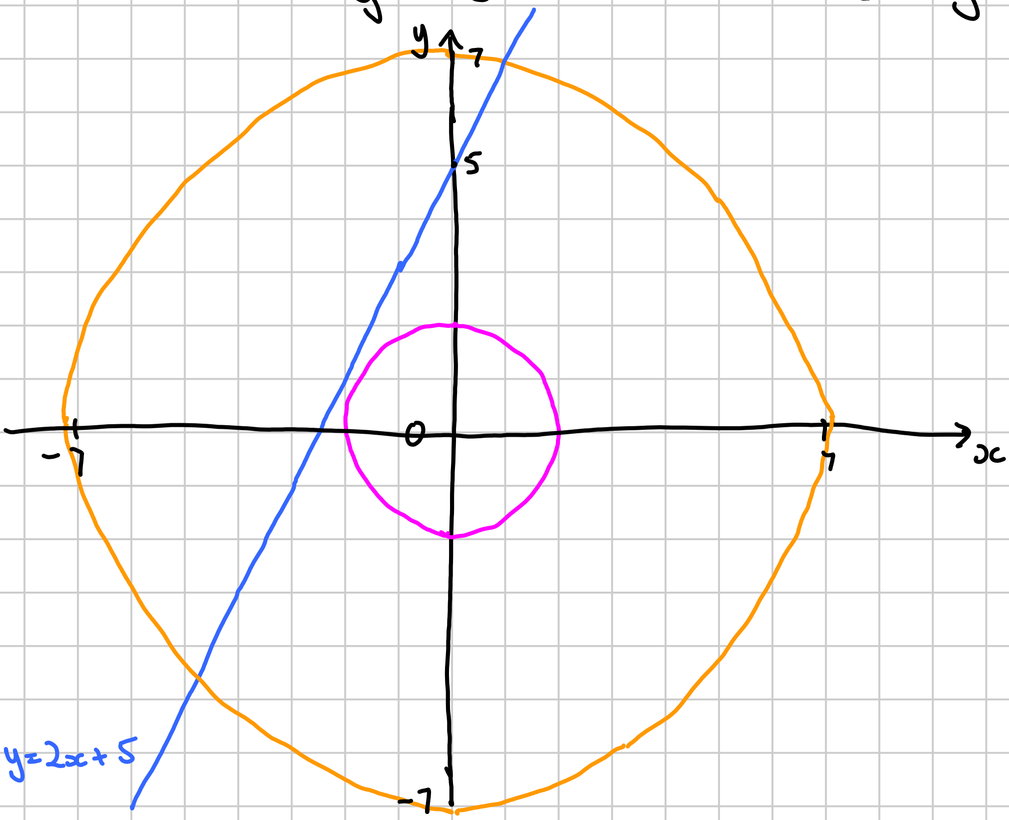
Centre is $(3, -4)$, radius is 5

p51 Ex 4A Q 2, 4, 6, 7

p59 Ex 4C Q 2, 3, 5, 8

p63 Ex 4D Q 1ac, 2ac, 3ae, 4, 5, 9, 10

(4) (a) Find the points of intersection of the circle $x^2 + y^2 = 50$ and the line $y = 2x + 5$



Solve simultaneously: substitute eqn of line into eqn of circle

$$x^2 + (2x+5)^2 = 50$$

$(2x+5)(2x+5)$

$$x^2 + 4x^2 + 20x + 25 = 50$$
$$5x^2 + 20x - 25 = 0$$
$$x^2 + 4x - 5 = 0$$
$$(x+5)(x-1) = 0$$

$x = -5$ or $x = 1$

(Subst into $y = 2x+5$:)

$y = -5$ $\qquad y = 7$

Points of intersection are $(-5,-5)$ and $(1,7)$

(b) Show that the line $y = 2x+5$ does not intersect the circle $x^2+y^2 = 4$

Try to find the points of intersection:

$$x^2 + (2x+5)^2 = 4$$

(working as above)

$$5x^2 + 20x + 25 = 4$$
$$5x^2 + 20x + 21 = 0$$

Discriminant: $b^2 - 4ac = 400 - 4 \times 5 \times 21$
$= -20$

Negative discriminant $\Rightarrow$ no real roots
$\Rightarrow$ line and circle do not intersect.

(c) The line $y = 2x+5$ is a tangent to the circle $x^2+y^2 = r^2$. Find the value of r and the point where the tangent touches the circle.

As before,
$$x^2 + (2x+5)^2 = r^2$$
$$5x^2 + 20x + 25 = r^2$$
$$5x^2 + 20x + (25 - r^2) = 0$$

For the line to be a tangent to the circle, there must be one point of intersection ie/ this equation must have one (repeated) root.

So $b^2 - 4ac$ must equal zero.

$$400 - 4 \times 5 \times (25 - r^2) = 0$$
$$400 - 500 + 20r^2 = 0$$
$$20r^2 = 100$$
$$r^2 = 5$$
$$\underline{\underline{r = \sqrt{5}}}$$

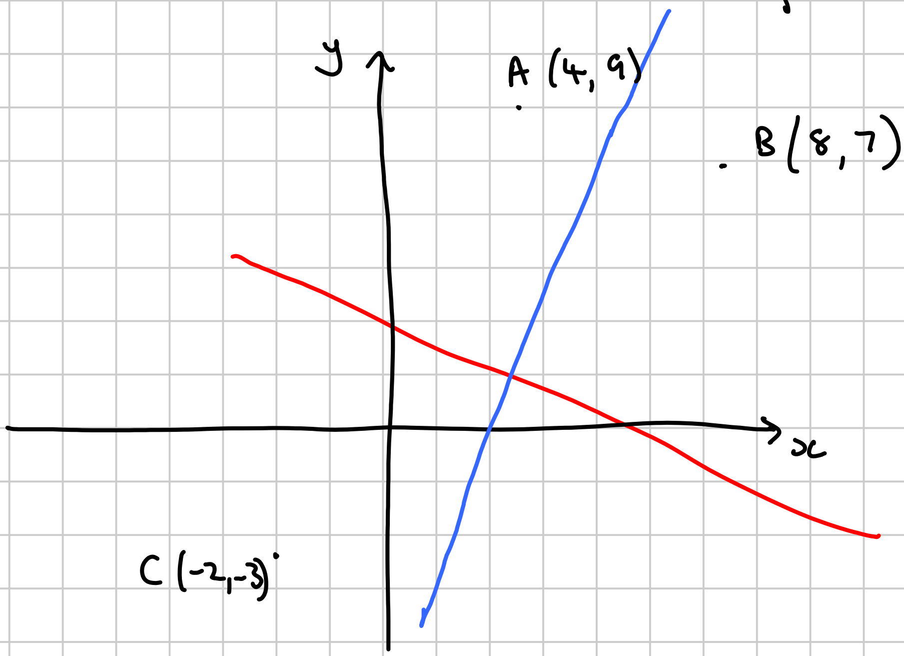
Put $r = \sqrt{5}$ into eqn above:

$$5x^2 + 20x + 20 = 0$$
$$x^2 + 4x + 4 = 0$$
$$(x+2)(x+2) = 0$$

$(y = 2x + 5)$ $\left.\begin{array}{l} x = -2 \\ y = 1 \end{array}\right\}$ line touches curve at $(-2, 1)$

p66 Ex 4E Q 1, 4, 6, 7, 9

⑤ The points A $(4, 9)$, B $(8, 7)$ and C $(-2, -3)$ lie on a circle. Find the equation of the circle.



Centre of circle must lie on perpendicular bisector of AB

Midpoint of AB is $\left(\frac{4+8}{2}, \frac{9+7}{2}\right) = (6, 8)$

Gradient of AB is $\frac{9-7}{4-8} = -\frac{1}{2}$

Gradient of perpendicular bisector $= 2$

Eqn of perpendicular bisector

$$y - 8 = 2(x-6)$$
$$y - 8 = 2x - 12$$
$$y = 2x - 4$$

Centre of circle must also lie on perp. bisector of AC

Midpoint of AC is $\left(\frac{4+-2}{2}, \frac{9+-3}{2}\right) = (1, 3)$

Gradient of AC is $\frac{9--3}{4--2} = \frac{12}{6} = 2$

Gradient of perp bisector is $-\frac{1}{2}$

Eqn of perp bisector is

$$y - 3 = -\tfrac{1}{2}(x-1)$$
$$2y - 6 = -x + 1$$
$$x + 2y = 7$$

Centre is where $y = 2x - 4$ ①

and $x + 2y = 7$ ② cross

Subst ① into ② $\Rightarrow$

$$x + 2(2x-4) = 7$$
$$5x - 8 = 7$$
$$5x = 15$$
$$x = 3$$
$$y = 2$$

Centre is $(3, 2)$

Radius is $\sqrt{(8-3)^2 + (7-2)^2} = \sqrt{5^2 + 5^2} = \sqrt{50}$

$\left(\sqrt{(4-3)^2 + (9-2)^2} = \sqrt{1^2 + 7^2} = \sqrt{50}\right)$

Equation is $(x-3)^2 + (y-2)^2 = 50$

p56 Ex 4B Q 5, 7, 9

p66 Ex 4E Q 1, 4, 6, 7, 9