EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Note Title 01/12/2011

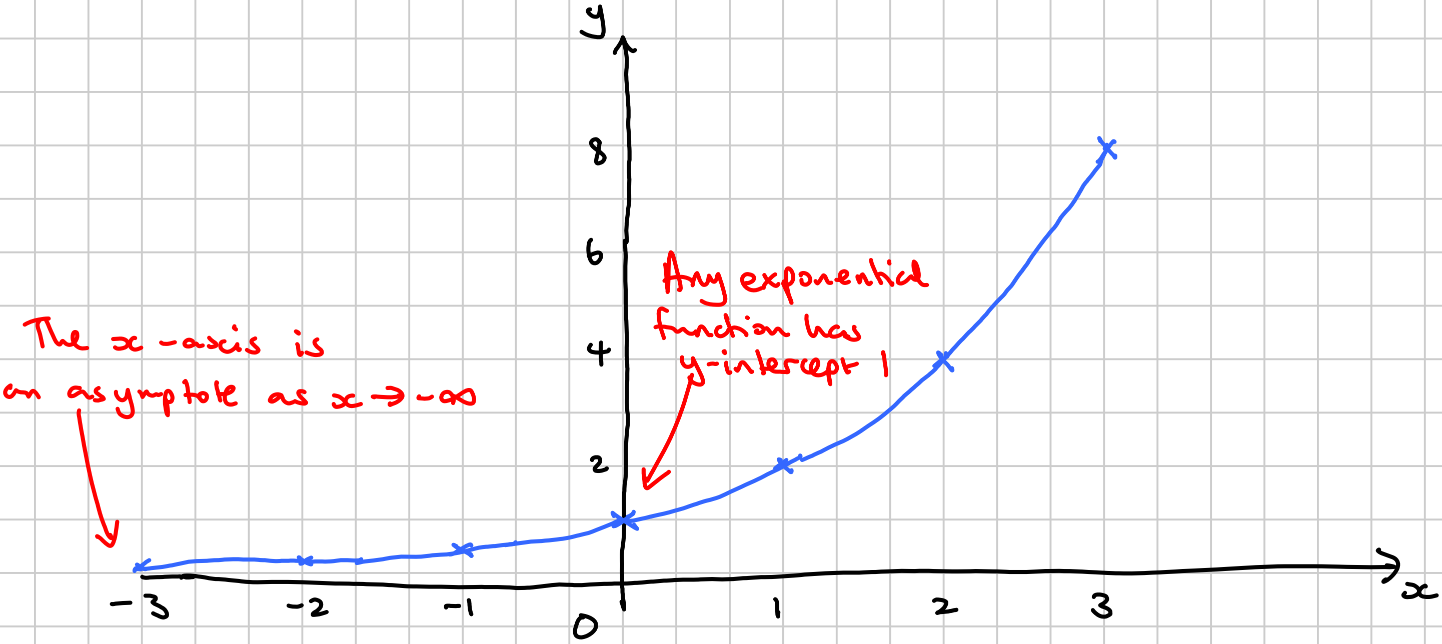
Exponential Functions

A function of the form $y = a^x$ (where a is a number) is an exponential function.

All exponential functions have the same basic shape :—

e.g. $y = 2^x$

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8



<u>Note</u> If $a = \frac{1}{2}$, $y = \left(\frac{1}{2}\right)^x$ is the same as $y = \frac{1}{2^x}$

or $y = 2^{-x}$

So the graph of $y = \left(\frac{1}{2}\right)^x$ is the reflection of $y = 2^x$ in the y-axis.

Logarithms

A logarithm is a power or index.

e.g. if $2^x = 32$, then $x = 5$ $(2 \times 2 \times 2 \times 2 \times 2 = 32)$

"the power of 2 which makes 32 is 5"

we abbreviate this as :

$$\log_2 32 = 5$$

More examples: $\log_2 8 = 3$

$\log_3 9 = 2$

$\log_2 \frac{1}{8} = -3 \iff 2^{-3} = \frac{1}{8}$

$\log_{10} 10 = 1$

$\log_a 1 = 0 \iff a^0 = 1$

$\log_4 2 = \frac{1}{2} \iff 4^{1/2} = 2$

$$\boxed{a^n = x \iff n = \log_a x}$$

p 40 Ex 3B Q 1–3

Laws of Logarithms

These are basically the laws of indices written in a different form.

First some examples with numbers:—

$2^3 = 8$ $2^5 = 32$	$\log_2 8 = 3$ $\log_2 32 = 5$
$2^3 \times 2^5 = 2^8$ $8 \times 32 = 256$	$\log_2(8 \times 32) = 8$ $= \log_2 8 + \log_2 32$
$a^m \times a^n = a^{m+n}$ $\iff$	$\boxed{\log_a(xy) = \log_a x + \log_a y}$
$\frac{a^m}{a^n} = a^{m-n}$ $\iff$	$\boxed{\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y}$
$(2^3)^4 = 2^{12}$ $8^4 = 4096$	$\log_2 4096 = 12$ $\log_2(8^4) = 12 = 4 \times 3$ $\log_2(8^4) = 4 \times \log_2 8$

$(a^m)^n = a^{mn}$ $\Longleftrightarrow$ $\boxed{\log_a(x^n) = n\log_a x}$

$\log_2 4096 = 12$

$\log_8 4096 = 4 = \frac{12}{3}$

$= \frac{\log_2 4096}{\log_2 8}$

$$\boxed{\log_b x = \frac{\log_a x}{\log_a b}}$$

Examples

① Expand:

(a) $\log\left(\frac{a^2 b}{c}\right) = \log(a^2 b) - \log(c)$

$= \log(a^2) + \log(b) - \log(c)$

$= 2\log(a) + \log(b) - \log(c)$

(b) $\log(\sqrt{ab}) = \log\left((ab)^{1/2}\right)$

$= \frac{1}{2}\log(ab)$

$= \frac{1}{2}\left[\log(a) + \log(b)\right]$

$= \frac{1}{2}\log(a) + \frac{1}{2}\log(b)$

(c) $\log_{10}(100x^2) = \log_{10} 100 + \log_{10} x^2$

$= 2 + 2\log_{10} x$

② Express as a single logarithm:-

(a) $3\log a - \frac{1}{2}\log b = \log a^3 - \log\sqrt{b}$

$= \log\left(\frac{a^3}{\sqrt{b}}\right)$

(b) $\log 54 - 2\log 3 = \log 54 - \log 9$

$= \log\left(\frac{54}{9}\right) = \log 6$

(c) $3\log_a 2 + 2 = \log_a 8 + 2\log_a a$

We need a log here so multiply by $\log_a a$ which is 1

$= \log_a 8 + \log_a (a^2)$

$= \log_a (8a^2)$

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p 43 Ex 3D Q (all)

Solving Equations

① $3^x = 20$

$x = \log_3 20$

$= 2.73 \quad (3 \text{ s.f.})$

② $4^{2x-3} = 32$

Take $\log_4$ of both sides

$\log_4(4^{2x-3}) = \log_4 32$

$(2x-3)\log_4 4 = \log_4 32 \qquad (\text{but } \log_4 4 = 1)$

$2x - 3 = 2.5$

$2x = 5.5$

$x = 2.75$

③ $5^{x+1} = 6^{x-1}$

Take $\log_5$ of both sides (could take $\log_6$ or log to any base)

$\log_5(5^{x+1}) = \log_5(6^{x-1})$

$(x+1)\cancel{\log_5 5} = (x-1)\log_5 6$

$x + 1 = x\log_5 6 - \log_5 6$

$1 + \log_5 6 = x\log_5 6 - x$

$= x(\log_5 6 - 1)$

$\dfrac{1 + \log_5 6}{\log_5 6 - 1} = x$

$x = 18.7$ (3 sf)

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p 43 Ex 3D Q (all)

Ex 3E Q 1 a e f g h

Q 2 a d e

Like this one

④ $2^{2x} - 2^{x+2} - 5 = 0$

Let $2^x = y$. Then $2^{2x} = 2^{x \times 2}$

$= (2^x)^2$

$= y^2$

And $2^{x+2} = 2^x \times 2^2$

$= 4y$

So the equation becomes $y^2 - 4y - 5 = 0$

$(y+1)(y-5) = 0$

$y = -1$ or $y = 5$

So $2^x = -1$ or $2^x = 5$

$x = \log_2(-1)$ [NOT POSSIBLE]

$\underline{\underline{x = \log_2 5}}$ (or $x = 2.32$ 3 sf)

[Note that we cannot have the log of a negative number.]

(5) Solve the equation

$$\log_4 2x + \log_{16} x = \log_4 54$$

$$\log_4 2x + \frac{\log_4 x}{\log_4 16} = \log_4 54$$

$$\log_4 2x + \tfrac{1}{2}\log_4 x = \log_4 54$$

$$\log_4 2x + \log_4 x^{1/2} = \log_4 54$$

[NB We CANNOT just 'get rid' of all the "$\log_4$"s at this stage – we need to reach a stage where we a SINGLE logarithm on each side.]

$$\log_4 2x\sqrt{x} = \log_4 54$$

Now we can say

$$2x\sqrt{x} = 54$$

$$x\sqrt{x} = 27$$

$$x^2 \times x = 729$$

$$x^3 = 729$$

$$\underline{x = 9}$$

Ex 3E Q 1bdfh, 2bd
3F Q 1bd, 3bc
3G Q 1, 2, 3, 4, 6, 8, 9, 11, 12