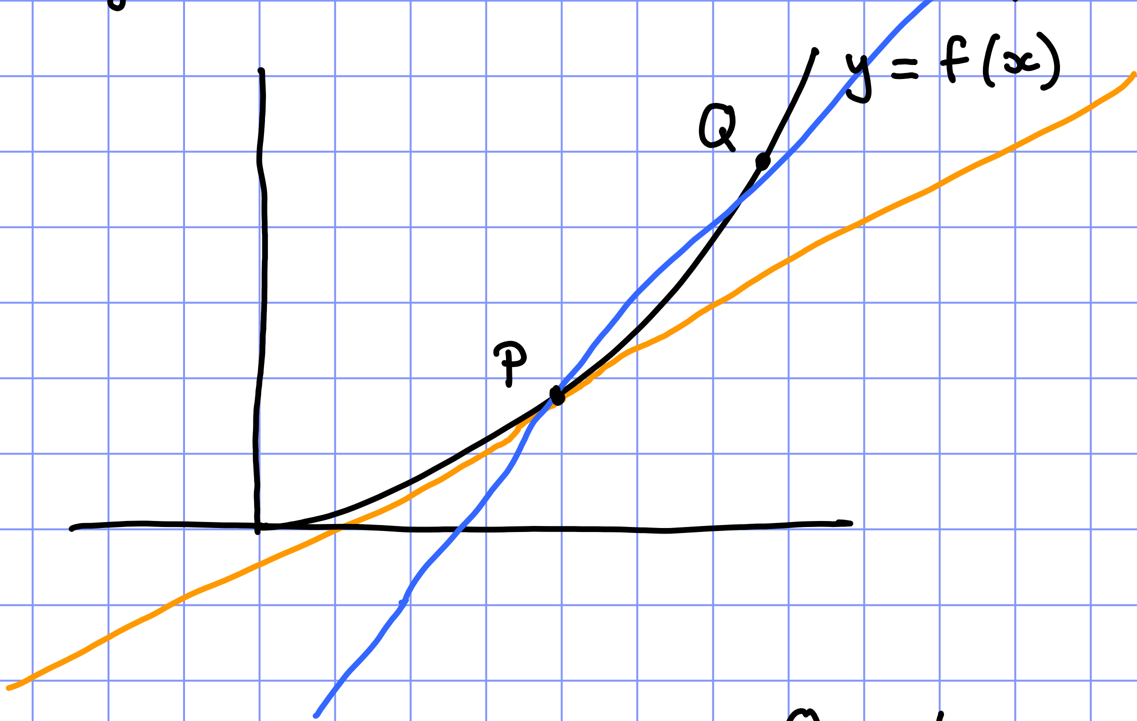
Differentiation

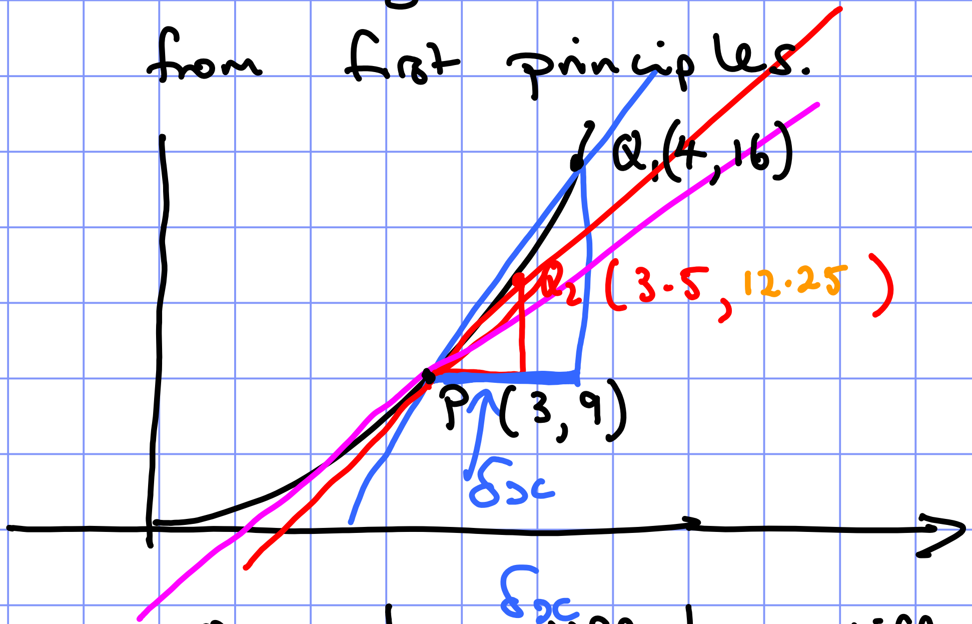
A straight line has the same gradient at every point on the line.
A curve has a different gradient at each point, which is equal to the gradient of the tangent at that point

In order to find the gradient of a tangent at a point P, we can find the gradient of a chord PQ.



As we move the point Q closer to the point P, the gradient of the chord PQ gets closer and closer to the gradient of the tangent. So the LIMIT of this sequence of gradients is the gradient of the tangent.

__Example 1__ Find the gradient of the curve $y = x^2$ at the point $(3, 9)$, from first principles.

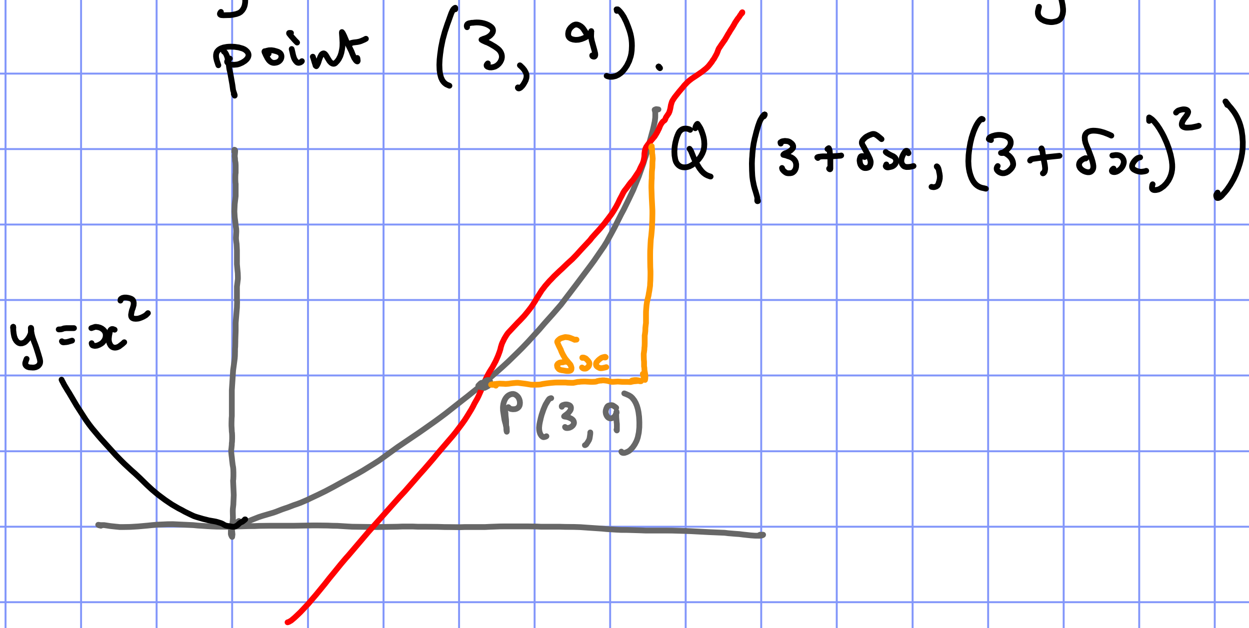


Q	δx x-diff	y-diff	PQ gradient
(4, 16)	1	7	7
(3.5, 12.25)	0.5	3.25	6.5
(3.1,)	0.1		6.1
(3.01,)	0.01		6.01
(3.001,)	0.001		6.001

Since the gradient is tending to a __limit__ of 6, we conclude that the gradient of the curve at the point (3, 9) is 6

We now try to express the process of finding the limit using algebra. To do this we express the difference in the x- direction between P and Q as a variable δx. (Note that this is a single variable - not $\delta \times x$.)

Example 1 (revisited) Find the gradient of the curve $y = x^2$ at the point $(3, 9)$.



gradient of chord PQ

$$= \frac{(3+\delta x)^2 - 9}{\delta x}$$

$$= \frac{\cancel{9} + 6\delta x + (\delta x)^2 - \cancel{9}}{\delta x}$$

$$= \frac{\delta x(6 + \delta x)}{\delta x} \quad *$$

$$= 6 + \delta x$$

gradient of curve at P

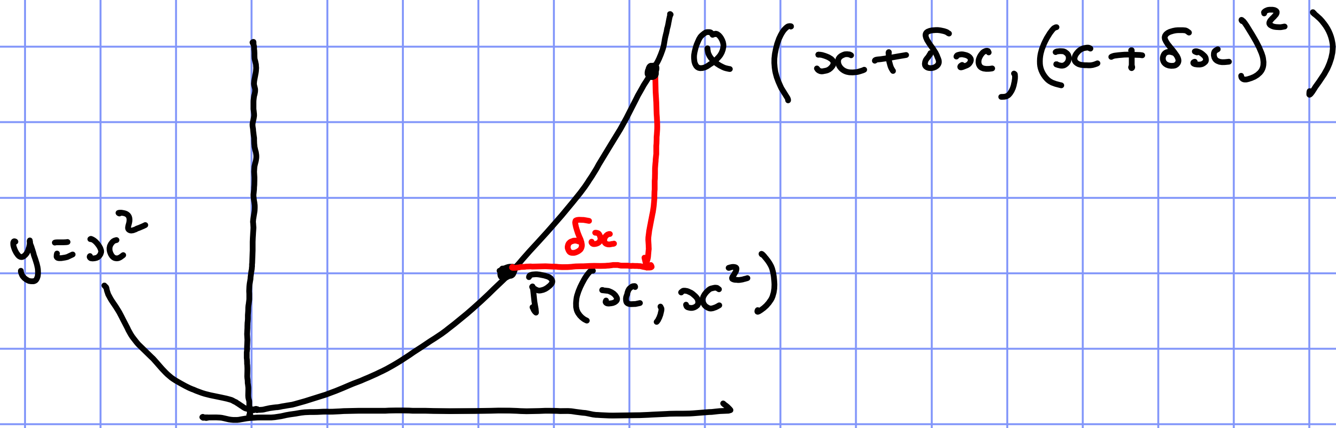
$$= \lim_{\delta x \to 0} (\text{gradient of PQ})$$

$$= \lim_{\delta x \to 0} (6 + \delta x)$$

$$= \underline{\underline{6}}$$

Now we wish to generalize this in order to find a FUNCTION which gives us the gradient at any point on the curve $y = x^2$.

Example 1 (again!)



$$\text{gradient at } P = \lim_{\delta x \to 0} \left(\frac{(x+\delta x)^2 - x^2}{\delta x} \right)$$

$$\boxed{\begin{array}{c} \delta x \times x \\ \neq \delta x^2 \end{array}}$$

$$= \lim_{\delta x \to 0} \left(\frac{\cancel{x^2} + 2x(\delta x) + (\delta x)^2 - \cancel{x^2}}{\delta x} \right)$$

$$= \lim_{\delta x \to 0} \left(\frac{\cancel{\delta x}(2x + \delta x)}{\cancel{\delta x}} \right)$$

$$= \lim_{\delta x \to 0} (2x + \delta x)$$

$$= \underline{\underline{2x}}$$

This function is called the DERIVATIVE FUNCTION (or just the DERIVATIVE) of the function $f(x) = x^2$.

The process of obtaining the derivative function is called DIFFERENTIATION

We write :-

$$f(x) = x^2 \quad \Rightarrow \quad f'(x) = 2x$$

or

$$y = x^2 \quad \Rightarrow \quad \frac{dy}{dx} = 2x$$

("the derivative of y with respect to x")

(" dy BY dx ")

or

$$\frac{d}{dx}\left(x^2\right) = 2x$$

("the derivative with respect to x of x^2 is $2x$")

Find the derivative function of

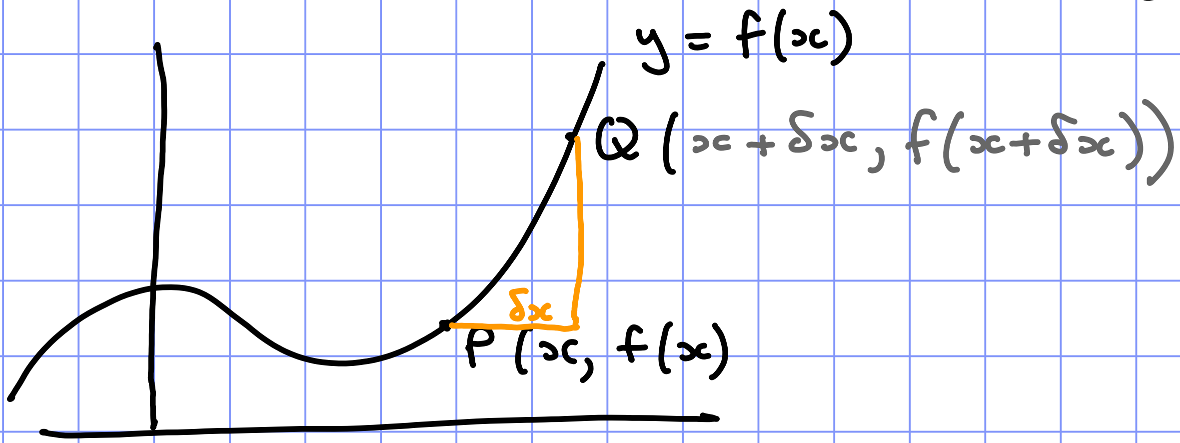
① $y = x^3$

② $y = \frac{1}{x}$

(Repeat the working of Example 1 (mark 3!) for $(x+\delta x)^3$

each of these curves).

The general formula for the derivative function of $f(x)$ is :—



$$\boxed{f'(x) = \lim_{\delta x \to 0} \left(\frac{f(x+\delta x) - f(x)}{\delta x} \right)}$$

Rules for differentiation

It can be shown that :

$$\boxed{\text{if } \quad y = x^n , \quad \text{then } \frac{dy}{dx} = n x^{n-1}}$$

(this is true for any n, positive or negative or fractional)

Also,

$$\text{if } \quad y = f(x) \pm g(x) \text{ then } \frac{dy}{dx} = f'(x) \pm g'(x)$$

if $y = k f(x)$ then $\frac{dy}{dx} = k f'(x)$

(where k is a constant)

(These two rules mean that differentiation is a LINEAR OPERATOR.
Other linear operators include
lim $\quad \Sigma \quad \int$ (integral))

Examples

① If $y = \frac{1}{x^4} + \sqrt{x}$, find $\frac{dy}{dx}$

$$y = x^{-4} + x^{1/2}$$

so $$\frac{dy}{dx} = -4x^{-5} + \frac{1}{2}x^{-1/2}$$

$$= \frac{-4}{x^5} + \frac{1}{2\sqrt{x}}$$

② If $f(x) = (x^2+3)(x^3+1)$ find $f'(x)$.

$$f(x) = x^5 + x^2 + 3x^3 + 3$$

so $$f'(x) = \underline{\underline{5x^4 + 2x + 9x^2 + 0}}$$

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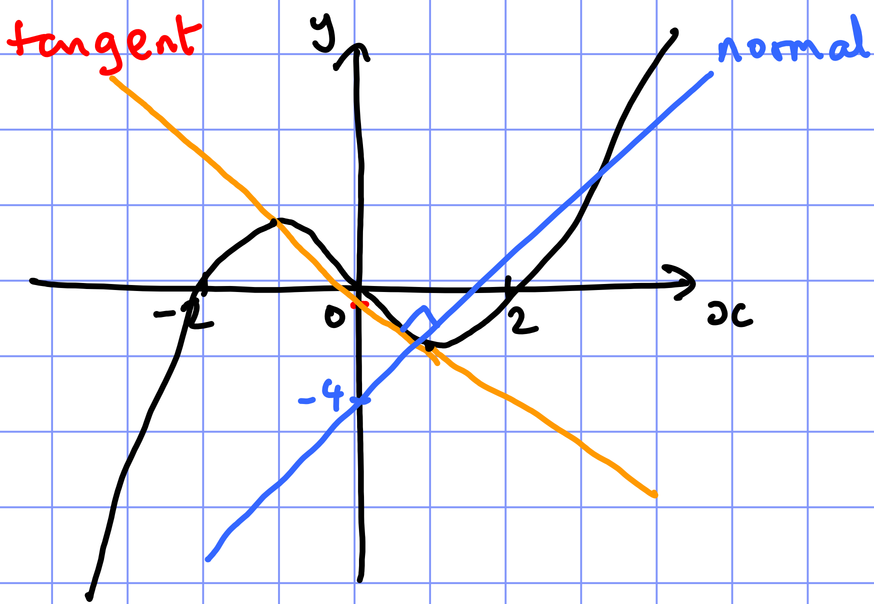
③ Find the gradient of the curve $y = x^3 - 4x$ at the point $(1, -3)$.
Hence find the equation of the tangent and the normal at that point.



$$\frac{dy}{dx} = 3x^2 - 4$$

At $x = 1$,

$$\text{gradient} = 3 \times 1^2 - 4 = -1$$

Equation of tangent is

$$y - (-3) = -1(x - 1)$$
$$y + 3 = -x + 1$$
$$\underline{y = -x - 2}$$

Gradient of the normal is 1

Equation of the normal is

$$y - (-3) = 1(x - 1)$$
$$y + 3 = x - 1$$
$$\underline{y = x - 4}$$

The second derivative

The derivative of the derivative is called the second derivative.

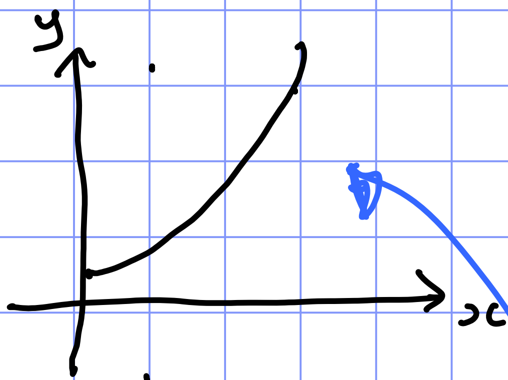
It is written $f''(x)$ or $\dfrac{d^2y}{dx^2}$

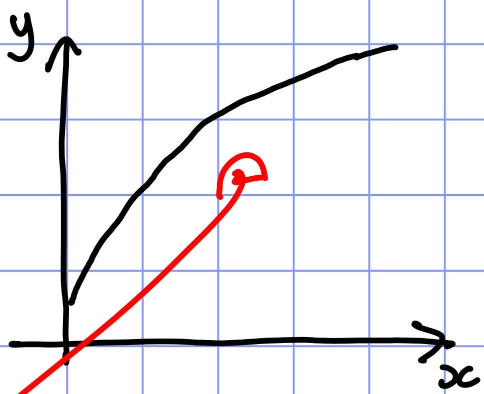
("d two y by d x squared")

Example If $y = 4x^2 - 3x + 2.$

then $\dfrac{dy}{dx} = 8x - 3$

and $\dfrac{d^2y}{dx^2} = 8$





$\dfrac{dy}{dx} > 0$

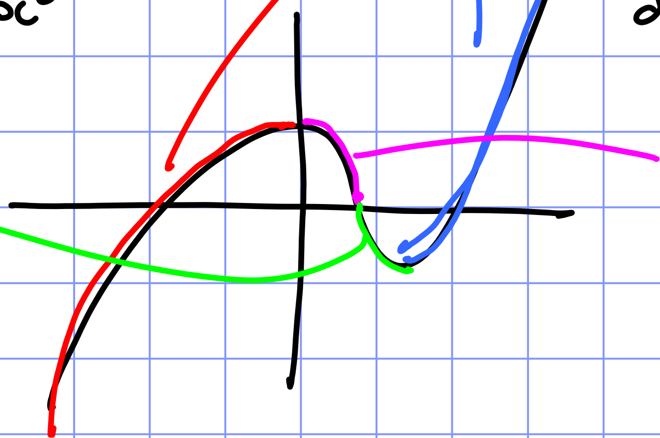
$\dfrac{d^2y}{dx^2} > 0$

$\dfrac{dy}{dx} > 0$

$\dfrac{d^2y}{dx^2} < 0$

$\dfrac{dy}{dx} < 0$

$\dfrac{d^2y}{dx^2} > 0$



$\dfrac{dy}{dx} < 0$

$\dfrac{d^2y}{dx^2} < 0$

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Ex 7F Q 3, 4, 5

Ex 7G Q 1, 2

Ex 7H Q 3, 5