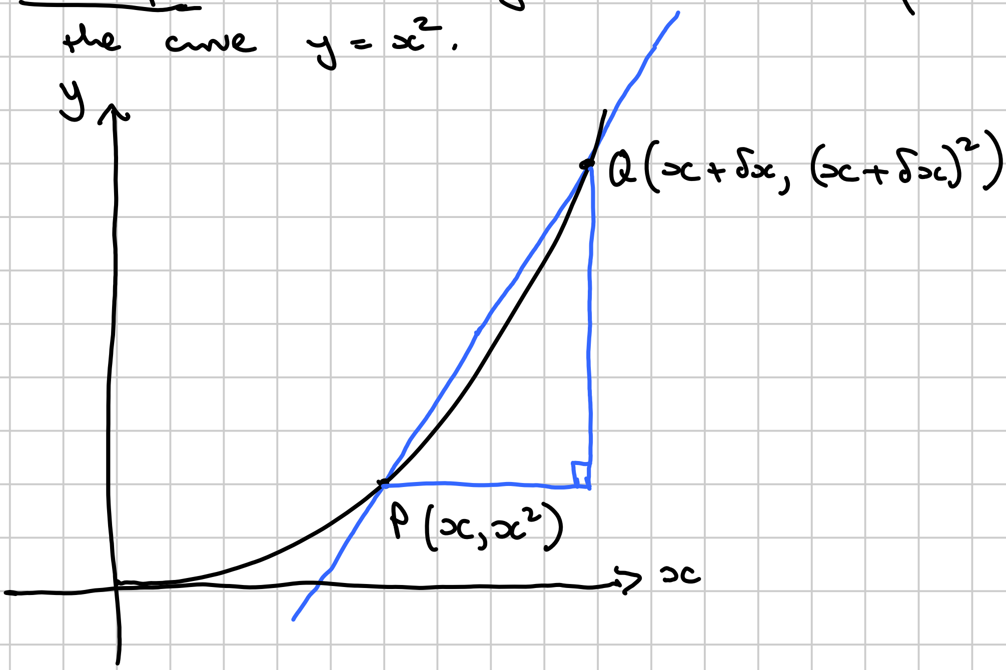
Differentiation

Note Title 07/11/2011

The gradient of a curve at a point P is equal to the gradient of the tangent at P.
It is defined as the LIMIT as $Q \to P$ of the gradient of a chord PQ where Q is another point on the curve.

Example Find the gradient at the point P (x, x^2) on the curve $y = x^2$.



$$\text{Gradient of } PQ = \frac{(x+\delta x)^2 - x^2}{\delta x}$$

$$= \frac{\cancel{x^2} + 2x\delta x + (\delta x)^2 - \cancel{x^2}}{\delta x}$$

$$= \frac{\delta x\,(2x + \delta x)}{\delta x}$$

$$= 2x + \delta x$$

$$\text{Gradient at } P = \lim_{\delta x \to 0} \left(\text{Gradient of } PQ \right)$$

$$= \lim_{\delta x \to 0} (2x + \delta x)$$

$$= 2x$$

The function $x \to 2x$ gives the gradient of the curve $y = x^2$ at any point (x, x^2) on the curve.

This function is called the DERIVATIVE FUNCTION (or just DERIVATIVE) of the function $f(x) = x^2$.

It can be written in two ways:–

$$f(x) = x^2 \Longrightarrow f'(x) = 2x$$

or

$$y = x^2 \Longrightarrow \frac{dy}{dx} = 2x$$

Generalising the working above for any curve $y = f(x)$, we find that

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \left(\frac{f(x+\delta x) - f(x)}{\delta x} \right)$$

Applying this rule to the function $y = x^n$, we can show that

$$\text{if } y = x^n, \quad \frac{dy}{dx} = nx^{n-1}$$

(This is valid for all values of n including negative and fractional indices.)

Also

$$\text{If } y = mx, \quad \frac{dy}{dx} = m$$

$$\text{If } y = c \quad \frac{dy}{dx} = 0$$

Also

$$\text{If } y = f(x) + g(x), \quad \frac{dy}{dx} = f'(x) + g'(x)$$

$$\text{If } y = kf(x), \quad \frac{dy}{dx} = kf'(x)$$

BUT

$$\text{If } y = f(x)\,g(x), \quad \frac{dy}{dx} \neq f'(x)\,g'(x)$$

$$\text{If } y = \frac{f(x)}{g(x)}, \quad \frac{dy}{dx} \neq \frac{f'(x)}{g'(x)}$$

Examples

① Find the gradient of the curve $y = 3x^3 - 4x^2 + 7x - 5$ at the point $(2, 17)$.

$$\frac{dy}{dx} = 3(3x^2) - 4(2x) + 7 - 0$$
$$= 9x^2 - 8x + 7$$

If $x = 2$, $\frac{dy}{dx} = \underline{\underline{27}}$

② Find the gradient of $y = (3x-2)(x^2+4x)$ at the points where the curve crosses the x-axis.

$\boxed{\text{NOTE: } \frac{dy}{dx} \text{ is NOT } (3)(2x+4)\,!}$

First write $y = 3x^3 + 12x^2 - 2x^2 - 8x$
$$= 3x^3 + 10x^2 - 8x$$
$$\frac{dy}{dx} = 9x^2 + 20x - 8$$

Curve crosses x-axis at $y = 0$

$$\Rightarrow (3x-2)(x^2+4x) = 0$$
$$x(3x-2)(x+4) = 0$$

$x = 0$ or $x = \frac{2}{3}$ or $x = -4$

$$\left(\begin{array}{l} 3x - 2 = 0 \\ 3x = 2 \\ x = 2/3 \end{array}\right)$$

If $x = 0$, $\frac{dy}{dx} = \underline{\underline{-8}}$

If $x = 2/3$ $\frac{dy}{dx} = \cancel{9} \times \frac{4}{\cancel{9}} + 20 \times \frac{2}{3} - 8$
$$= 4 + 13\tfrac{1}{3} - 8$$
$$= \underline{\underline{9\tfrac{1}{3}}}$$

If $x = -4$ $\frac{dy}{dx} = 9 \times 16 + 20 \times (-4) - 8$
$$= \underline{\underline{56}}$$

③ Find the points on $y = 4x + \frac{2}{x}$ where the gradient is -4.

First write

$$y = 4x + 2x^{-1}$$

$$\frac{dy}{dx} = 4 + 2(-1x^{-2})$$

$$= 4 - \frac{2}{x^2}$$

We want

$$4 - \frac{2}{x^2} = -4$$

$$4x^2 - 2 = -4x^2$$

$$8x^2 - 2 = 0$$

$$8x^2 = 2$$

$$x^2 = 1/4$$

$$x = 1/2 \quad \text{or} \quad -1/2$$

If $x = 1/2$, $y = 4 \times \frac{1}{2} + \frac{2}{1/2} = 6$

$x = -1/2$, $y = 4 \times (-1/2) + \frac{2}{-1/2} = -6$

Points are $(\frac{1}{2}, 6)$ and $(-1/2, -6)$

④ Find $\frac{dy}{dx}$ if $y = \frac{x^2 - 3}{\sqrt{x}}$

Simplify first:

$$y = x^{-1/2}(x^2 - 3)$$

$$= x^{3/2} - 3x^{-1/2}$$

$$\frac{dy}{dx} = \frac{3}{2}x^{1/2} - 3\left(-1/2\, x^{-1 1/2}\right)$$

$$= \frac{3}{2}x^{1/2} + \frac{3}{2}x^{-3/2}$$

P 112 Ex 7C Q 1a, 2f, 3, 4, 5

P 115 Ex 7E Q 1, 2bc

Finish for homework

(5) (a) Find the equation of the tangent to the curve $y = x^3 - 4x + 3$ at the point with x-coordinate 3.

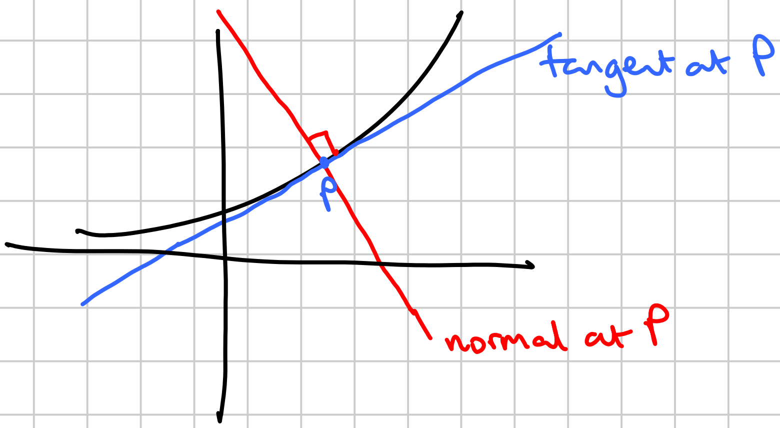
y-coord: $y = 3^3 - 4 \times 3 + 3$
$= 18$ ie, $(3, 18)$

$$\frac{dy}{dx} = 3x^2 - 4$$

At $(3, 18)$, gradient $= 3 \times 3^2 - 4 = 23$

Eqn of tangent is
$$y - 18 = 23(x - 3)$$
$$y - 18 = 23x - 69$$
$$\underline{\underline{y = 23x - 51}}$$

(b) Find the equation of the <u>normal</u> to the curve at the point $(3, 18)$



the <u>normal</u> to a curve is a straight line <u>perpendicular</u> to the curve at a given a point.

Gradient of normal $= \dfrac{-1}{\text{gradient of tangent}}$
$$= -\frac{1}{23}$$

Equation of normal is
$$y - 18 = -\frac{1}{23}(x - 3)$$
$$23y - 414 = -x + 3$$
$$\underline{\underline{x + 23y - 417 = 0}}$$

p119 Ex 7H Q 1bcd, 2, 3, 4