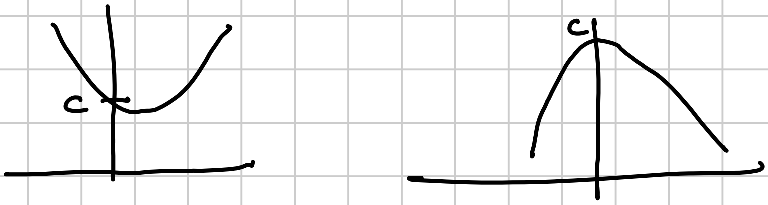
Quadratic Expressions and Equations

A quadratic expression is of the form

$$ax^2 + bx + c \qquad (a \neq 0)$$

Its graph is a parabola



$a > 0$ $a < 0$

It can be useful to write the expression in other ways:—

① **Factorise**

e.g. $3x^2 - 13x - 10$

($3 \times -10 = -30$ factors of -30 which add to -13 are -15 and 2.)

Factorise in pairs

$$= \underbrace{3x^2 - 15x} + \underbrace{2x - 10}$$

$$= 3x(x-5) + 2(x-5)$$

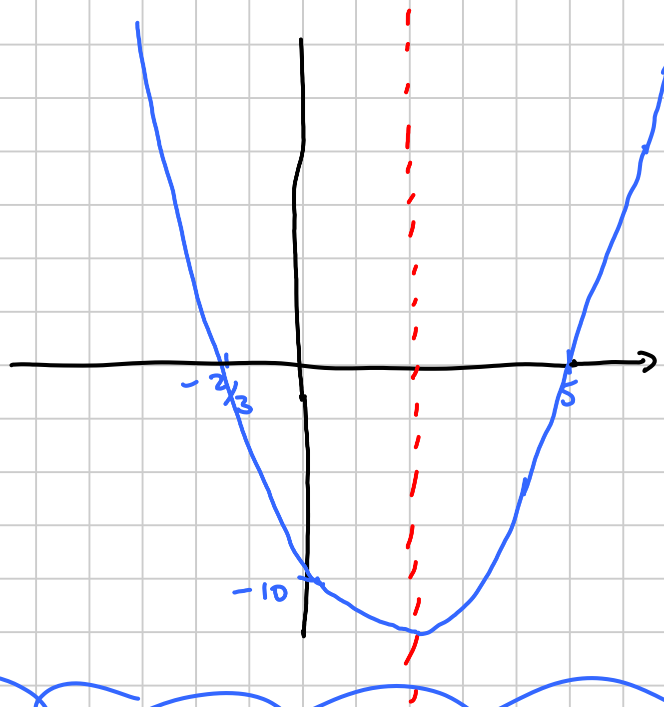
$$= (3x+2)(x-5)$$

From this we can solve $3x^2 - 13x - 10 = 0$ by saying

$$3x + 2 = 0 \quad \text{or} \quad x - 5 = 0$$

$$x = \frac{-2}{3} \quad \text{or} \quad x = 5$$

And this shows where the graph of $y = 3x^2 - 13x - 10$ crosses the x-axis:



p36 Ex 3.1 Q 1hio, 2, 3cd, 4ab, 5abcde

② Completing the square

Some quadratics factorise as a perfect square

e.g. $x^2 + 10x + 25 = (x+5)^2$

If the quadratic is not a perfect square we can make an adjustment

e.g $x^2 + 6x + 4$

(Halve coefficient of x : $\frac{1}{2}$ of $6 = 3$
Now square this : $3^2 = 9$)

$$= \underbrace{x^2 + 6x + 9}\ \underbrace{-\ 9 + 4}$$

$$= \underline{\underline{(x+3)^2 - 5}}$$

Now we can use this to :

- Find the minimum value of $x^2 + 6x + 4$

The square of a number is always non-negative
So $(x+3)^2 \geq 0$

$$\underset{(\geq -5)}{(x+3)^2} - 5 \geq \underset{(-5)}{-5}$$

Min value of $x^2 + 6x + 4$ is -5 and occurs when $x = -3$.

- Solve

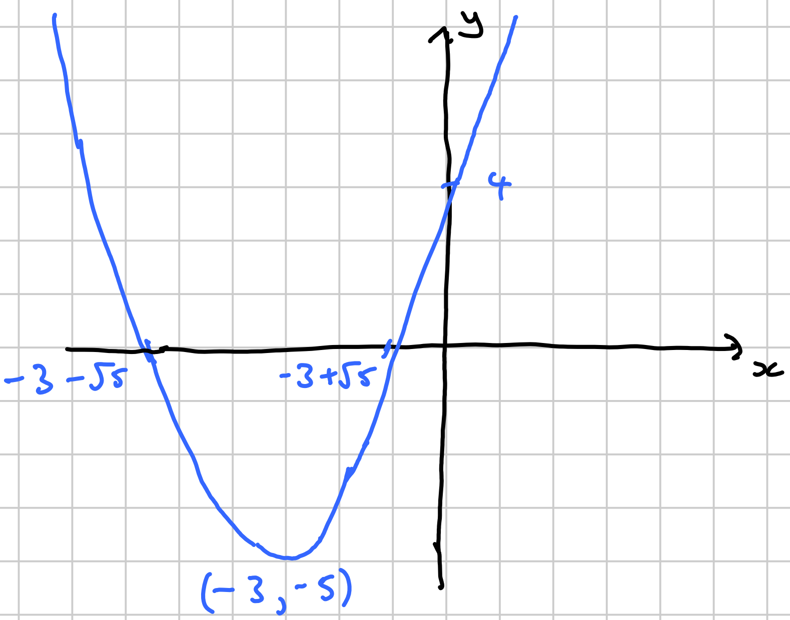
$$\begin{aligned} x^2 + 6x + 4 &= 0 \\ (x+3)^2 - 5 &= 0 \\ (x+3)^2 &= 5 \end{aligned}$$

(Square root both sides, remembering the $\pm$ sign)

$$\begin{aligned} x + 3 &= \pm\sqrt{5} \\ x &= \underline{-3+\sqrt{5} \text{ or } -3-\sqrt{5}} \end{aligned}$$

(leave answers like this unless told otherwise)

- Sketch the graph of $y = x^2 + 6x + 4$



More examples

① Write $x^2 - 8x + 20$ in the form $(x+q)^2 + r$

Hence:

(a) Find the min value of $x^2 - 8x + 20$

(b) Explain why $x^2 - 8x + 20 = 0$ has no solutions

(c) Sketch the graph of $y = x^2 - 8x + 20$.

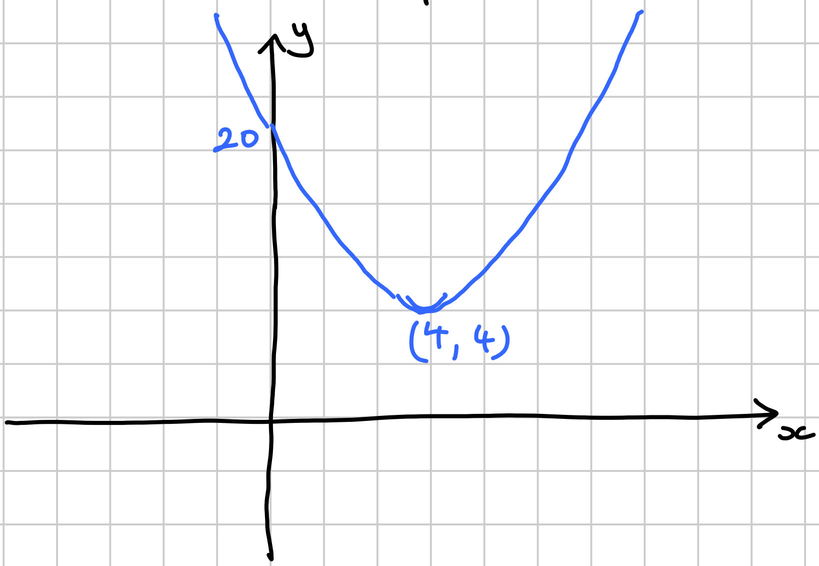
($\frac{1}{2}$ of $-8 = -4$, $(-4)^2 = 16$)

$$\begin{aligned} &x^2 - 8x + 16 - 16 + 20 \\ &= (x-4)^2 + 4 \end{aligned}$$

(a) Min value is 4, and occurs when $x = 4$

(b) Since $x^2 - 8x + 20$ cannot be less than 4, it can't be equal to 0.

(c)



② Write $3x^2 - 13x - 10$ in the form $p(x+q)^2 + r$

$$3\left(x^2 - \frac{13}{3}x - \frac{10}{3}\right)$$

(Half of $-\frac{13}{3}$ is $-\frac{13}{6}$, and $\left(-\frac{13}{6}\right)^2 = \frac{169}{36}$)

$$= 3\left(\underbrace{x^2 - \frac{13}{3}x + \frac{169}{36}} - \frac{169}{36} - \frac{10}{3}\right)$$

$$= 3\left(\left(x - \frac{13}{6}\right)^2 - \frac{169}{36} - \frac{120}{36}\right)$$

$$= 3\left(x - \frac{13}{6}\right)^2 - \frac{289}{12}$$

p 40 Ex 3.2 Q 2, 3bcd, 4, 5, 6

by Monday.

③ The quadratic formula and the discriminant

To solve the general quadratic equation

$$ax^2 + bx + c = 0$$

(Divide by a) $$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

(Half of $\frac{b}{a} = \frac{b}{2a}$, $\left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2}$)

$$\underbrace{x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}} - \frac{b^2}{4a^2} + \frac{c}{a} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

($\sqrt{\ }$ both sides) $$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

$\left(\sqrt{\frac{p}{q}} = \frac{\sqrt{p}}{\sqrt{q}}\right)$ $$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$$

$$\boxed{x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}}$$

which is the quadratic formula

The expression '$b^2 - 4ac$' is called the DISCRIMINANT of the quadratic equation.

If $b^2 - 4ac < 0$,
- the equation $ax^2 + bx + c = 0$ has no real roots ('root' means 'solution')
- the graph $y = ax^2 + bx + c$ does not cross the x-axis.

If $b^2 - 4ac = 0$,
- the equation has <u>one</u> repeated root $x = \frac{-b}{2a}$
- the expression $ax^2 + bx + c$ is a perfect square.

- the x-axis is a <u>tangent</u> to the graph, ie, the graph touches the x-axis at $x = \frac{-b}{2a}$.

If $b^2 - 4ac > 0$

- the equation has two distinct roots
- the graph crosses the x-axis twice

<u>Examples</u>

(1) For what values of k does the equation

$$x^2 - 4kx + k = 0$$

have a repeated root?

We want
$$b^2 - 4ac = 0$$
$$(-4k)^2 - 4 \times 1 \times k = 0$$
$$16k^2 - 4k = 0$$
$$4k(4k - 1) = 0$$
$$4k = 0 \quad \text{or} \quad 4k - 1 = 0$$
$$k = 0 \quad \text{or} \quad k = \frac{1}{4}$$

(2) Prove that $x^2 + (k-3)x - 2k = 0$ has real roots for all values of k.

$$\begin{aligned} b^2 - 4ac &= (k-3)^2 - 4 \times 1 \times (-2k) \\ &= k^2 - 6k + 9 + 8k \\ &= k^2 + 2k + 9 \\ &= k^2 + 2k + 1 - 1 + 9 \\ &= (k+1)^2 + 8 \\ &\geqslant 8 \end{aligned}$$

So $b^2 - 4ac$ is always +ve. Therefore the equation always has real roots.

p46 Ex 3.3 Q 2hi, 3acgh, 4ab, 6, 7, 11, 12